Curriculum Units by Fellows of the Yale-New Haven Teachers Institute
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Developing Word Problems: A Student's Task

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Purpose of Unit

To use the accumulated knowledge of word problems and the processes that are needed to solve them, to synthesize original, multi-step word problems that require thought and deduction.

Introduction and Rationale

During their stint in third grade, the students in the New Haven Public schools are expected to have proficiency in many areas within the realm of mathematics. They must first be proficient in their basic addition and subtraction facts and then upon that foundation much can be built. Multiplication and division are introduced during the course of this year as well as basic geometry. While the students venture through all of those operations, there is always one constant that reappears throughout it all. That constant is of course, the word problem.

When they enter the third grade, students are expected to have certain mathematical skills in place. They are expected to have mastered one-digit addition and subtraction without regrouping (facts to 18), as well as decoding and solving word problems. Ideally students would have enough knowledge and experience with word problems to understand the steps in which to solve them. These statements of course are the ideal, but to an educator, it becomes evident immediately within a classroom, that the ideal is not always reality. The reality is that students will enter a grade and be at varied levels, thus making a heterogeneously grouped classroom. Since every student is an individual and has individual needs, the teacher's job is to first assess the student's abilities and then differentiate the instruction according to his/her needs.

Within the domain of the word problem, the students have high expectations placed upon them. They must first decipher the vocabulary of the word problem while actively thinking about what the words involved mean. This can be done by reading the problem through carefully first without having an assumption of which function to use. An immediate assumption of which function to use can lead to an inadequate reading of the problem. Having sifted through the wording, the student must then identify what the end result of the problem is and what the question is asking. In many problems, the answer cannot be found unless a series of steps are completed and many students find themselves challenged by that process. It is the teacher's duty to teach students how to tackle word problems through modeling, analytical group discussions, and much guided practice. It is after that in which students can be challenged to use their own knowledge of the methods and strategies to independently solve word problems. During teacher instruction, classroom discussions should be encouraged. Discussions on word problems can aid in a student's understanding of the process of solving and deciphering problems, and can also be insightful for a teacher. When a student verbally expresses the processes used, he/she must be confident in the methods he/she utilized and must really think about how he/she arrived at the solution in order to articulate it. Teachers and other students can benefit by listening to an oral explanation. Teachers can understand the thought process of the student and pinpoint the area in which students are experiencing trouble. Other students can hear the methods their peers used to attack the problem and use those strategies to aid them if they encounter problems. In some cases, an explanation in "student lingo" can help a student more than an explanation given by a teacher. Discussions also allow the students to experience metacognition, or thinking about their own thinking. By doing this they can examine their processes to identify which are essential to solving the problems at hand.

In some instances the final step in a teacher's assessment of a student's proficiency of solving word problems is found by giving the student a problem and then examining the processes used and then the answer to the problem itself. For the sake of this unit on the other hand, this is not the last the student will see of the word problem. The goal of this unit is for students to use their accumulated knowledge of word problems and the processes that are needed to solve them, to synthesize their own complex, multi-step word problems.

The series of lessons within the unit are meant to teach students about word problems and to also immerse them in exploration and discussion. The reasoning behind this is to allow the students to become more familiar with the process of solving the problems and explaining how they arrive at their conclusions. Since scaffolding is the process of building upon information the students have previously learned, the lessons are designed to begin with simpler problems that then progressively and increase in difficulty.

It is also essential to peer into the psychological and developmental aspects of the students when it comes to mathematics. Piaget had strong views on mathematical word problems. There are great intellectual expectations that are placed upon students but are not facilitated within the classroom, thus leading to failure in mathematics. It is through the exploration of his writings that the realization of how reasoning in problem solving is often overlooked or misconstrued due to the inexperience of students. When the complexity of the problem increases some students will abandon logical thinking and practical methods to decipher the problem out of frustration. That then results in the students using incorrect operations to get a numeric answer. Piaget suggests working backwards when approaching problems by excluding the numerical values to have a narrow focus on the words themselves. After the numeric distractions have vanished, students can deductively reason what operation should be utilized and then the numbers can reappear in the problem.1

The rationale for creating this unit is to allow students to create and be actively involved in their own learning. The world problem is intimidating for some students and with that frame of mind they are hindered in their abilities. The goal is to take that scare-factor out of the word problem and have students feel confident to master solving them and then become fluent enough to create them. The process of learning must also be a process of doing. A student cannot fully comprehend unless he or she is immersed in doing. The formation of problems connects to this in such a way that students must be able to do and create a work in which others must also be able to understand and explore. It is then, of course, the teacher's duty to provide those opportunities that will involve the students in actively creating and synthesizing.2

My desire to enable student synthesis of quality word problems was inspired by Benjamin Bloom and the educational taxonomy that he created. The following are the stages within Bloom’s taxonomy and a brief explanation of what they each encompass taken from “The Learning Skills Program” website that adapted the taxonomy from, Bloom, B.S. (Ed.) (1956) Taxonomy of educational objectives: The classification of educational goals: Handbook I, cognitive domain. New York ; Toronto: Longmans, Green:

- Knowledge: observation and recall of information.
- Comprehension: understanding of information and grasping the meaning.
- Application: utilizing information as well as methods, concepts and theories.
- Analysis: seeing patterns, recognition of hidden meanings and identification of components.
- Synthesis: using old ideas to create new ones, generalizing from given fact, and relating knowledge from several areas.
- Evaluation: comparing and discriminating between ideas, assessing value of theories, and making choices based on a reasoned argument.3

The beginning stages of his system focus on memorization and the application of knowledge. The latter stages delve into the more complex thought processes in which students must fully understand the topic and then create a work that applies that prior knowledge, and proves the proficiency and knowledge of the topic. Synthesis is the second to last stage (before evaluation) that requires students to demonstrate activities that exhibit their knowledge and skills in certain areas. Unless students can show their skills through synthesis, they cannot be deemed fully proficient in the skill they are practicing.4 In the context of this unit, the knowledge portion would be the setting up of mathematical equations and deciphering the solutions to those, and synthesis would be their own creation of the problem.

Finally, the process of students generating their own word problems can also be used as an assessment by the teacher. The teacher can judge by student work if the student is fluent and proficient in the objectives of the lessons. There are two types of assessment that can be done as a result of the lessons contained in this unit. The first type is a performance-based assessment where the student generates a product and is then evaluated on it. The other assessment is process-focused which will prove to be an integral part to this unit. Process-focused assessment concentrates on not just what the answer is, but how the student arrives at that conclusion.5 Finding out how a student arrived at an answer is just as important (on some occasions more important) than the answer is. By assessing the process in which the student goes through, it can be determined if errors are just computational or if they are errors in reasoning and abstract operations.

New Haven Public Schools Mathematics Standards

In reading, evaluating, solving, and creating the problems incorporated in this unit student will be able to complete the following:

1.0 - Students will work comfortably and confidently within the real number system and operations.

> 1.0.b. - Students will add, i.e. join things together, increase.
> *1.0.d.* - Students will multiply, i.e. use repeated addition, count by multiples, combine things that come in groups.
> *1.0.f.* - Students will analyze problem situations and contexts in order to figure out when to add, subtract, multiply and divide

5.2 - Students will participate in the formulation of problems.

> 5.2.a. - Students will make decisions about the approach, materials, and strategies to use.
> *5.2.b.* - Students will use previously learned strategies, skills, knowledge, and concepts to make decisions.
> *5.2.c.* - Students will use strategies, such as using manipulatives or drawing sketches, to model problems.
> *5.2.e* - Students will extract pertinent information from situations and figure out what additional information is needed.

5.3 - Students will make the basic choices involved in planning and carrying out a solution.

> 5.3c. - Students will solve problems in ways that make sense and explain why these ways make sense, e.g. defend the reasoning, and explain the solution.
> *5.3.d.* - Students will write problems to a given number sentence.

5.4 - Students will move beyond a particular problem by making connections, extensions, and/or generalizations.

5.4.a. - Students will explain a pattern that can be used in similar situations.
5.4.b. - Students will explain how the problem is similar to other problems they have solved.

Unit Objectives

Students will:

- Utilize basic operations (addition, subtraction, multiplication, and division) to solve equations.
- Solve word problems with one step and also with multiple steps.
- Defend their solutions, giving justifications for solutions and methods leading to the solutions.
- Make connections between problems, noting the significance between inverse operations.
- Generate one step, and multiple step word problems based on equations.

These objectives will be fulfilled through whole class interactions, guided practice, and independent practice.

Whole Class Interaction

The first step to the final goal of student generated problems is teacher instruction. The teacher must model the process and explain the concepts explicitly.

This portion of the lesson is teacher directed and students are guided to discover what methods and strategies will aid them in solving word problems. Discussion is a focal point in this portion because the ultimate goal is for the students to grasp the knowledge of how to solve word problems. By discussing the possibilities, teacher and students can share their ideas and concerns for each set of problems. This class brainstorming is meant to help the students, not hinder their development. By making this a non-threatening environment where all ideas are accepted and evaluated, no answer or reason given by a student is stupid. Students need to take risks within the classroom and it is the teacher's responsibility as a facilitator for learning, to make every student feel comfortable enough to share his/her own ideas. It also should be noted that to lead a discussion, the teacher must be comfortable with the topic and evaluate the problems personally in order to be prepared to share his/her own reasoning. Word problem modeling must not only occur in a pencil-paper type environment, but also in the area of discussion. The teacher should set the stage for the students, establishing a comfort zone in which ideas can be expressed and opinions shared. Once this has been modeled, the teacher can integrate the students into the conversation, giving assistance when necessary.

It is necessary for students to not only know how to perform basic operations such as addition, subtraction, multiplication and division, but also to know that the operations are connected to each other. They must realize that addition and subtraction are inverse operations, meaning if you know that 1 + 1 = 2, then 2 - 1 = 1. The same principle can also be shown with higher numbers to show that if you know that 734 + 89 = 823, then 823 - 89 = 734. To be more exact, a given number (from any number) is the inverse to adding the given number: (a + b) - b = a. Multiplication and division are inverse operations as well, so that if: 3 x 5 is equal to 15 then 15 / 3 must then equal 5.

The following problems are a model for the teacher to follow and should walk the students through them. They are arranged in a Step Chart that includes the word problem (along with its inverse problem), the givens in the problem, steps to solve, and the rationale behind the methods used to solve it. The problems are arranged in this manner so the students become used to the format that they will later use when devising their own problems. The format also allows the students to lay out the information methodically to aid in their journey to the solution. Also note that the activity of analyzing word problems can be carried further by incorporating more complex problems based on the needs and skills of the students and should not be limited by the problems given within this unit.

Before each set of problems, a rationale for each is provided. This will outline the reasoning behind the ordering of the problems and the format they are in. In some rationales the term “basic” is utilized. This term indicates facts that a third grade student should, in most instances, be able to solve due to prior instruction and lessons. The goal of these word problems are not to stump the students on the equations, but to teach them how to read the problem, find out what the problem is asking for, solve it, and then be able to generate their own. To use extremely complex equations would defeat the purpose of the lesson and only frustrate the student.

It is also important to note that following work is the teacher’s preparatory work. Since the teacher must be familiar with how to solve the problem and also must be comfortable with explaining the routes to the solution, this framework should be used as a teacher’s guide before teaching the lesson. The teacher’s rationales and connections made trying to solve the problem may not always be identical to those of the students. The explanations given here are for the teacher model. The students are not asked to, nor are expected to, give the explanations below verbatim. What the teacher’s expectation should be is to have a class that is actively listening and adding their own ideas when they are inspired to. As long as the students are participating, discussing, and being led in the right direction, then the given rationale can take any form and go any route.

Problems A.1. through A.11.

Intended for students on-grade level

The problems listed for students on grade level increase in difficulty as they progress. It should be noted that the teacher should focus on each type of problem and may use the given problems. It is a strong possibility that a student may require more practice, so the teacher should then create his/her own parallel set of problems for each type before in fact moving on to the next set. This will allow students to practice each type until they are able to master the skills.

A.1. & A.2. skills: Addition and subtraction

The rationale behind this first set of problems is that they are formulated from basic addition and subtraction facts. The equations require regrouping of two-digit numbers, a skill in which third grade students learn the first few months of the year. Students appear to be more comfortable with addition, so A.1. is an addition problem. It is important for students to feel a level of success when then are learning new concepts and performing tasks. That confidence will then aid them in more complex problems, which they can then tackle without a high level of frustration. A.2. requires the inverse operation of A.1. to solve and when students become aware of the connection, the method to solve the second problem will become clear to them. The problems in Appendix A are one-step problems that require only one equation to arrive at the solution and belong to the same "addition-subtraction family."

A.3. & A.4. skills: Multiplication and division

The rationale behind A.3. and A.4. stems from the fact that third grade students are introduced to multiplication and division during the second half of the year. They practice one-digit multiplication, and are able to divide two-digit numbers by one-digit numbers evenly (without a remainder). A.3. and A.4. have the students practicing those skills. They must also get used to all types of vocabulary within the word problem and realize that when certain words appear, such as: times, groups of, equal parts, then multiplication and division are most likely to follow. A.3. and A.4. in Appendix B are still one-step problems that just deal with different operations than those that preceded.

A.5. & A.6. skills: Multiplication, division, addition, subtraction, and multiple steps

The next set of problems are the more complex that those before and require multiple steps. A tendency of some students when they encounter a word problem is to just add all of the numbers that appear. This is a habit that we as teachers try desperately to break because it is when they come upon problems like these that the students become lost. It also indicates a need for careful analysis in a while class discussion. A.5. starts with a basic multiplication equation and then an equation that requires basic regrouping. A.6. then requires basic subtraction with regrouping and then a basic division equation. A.5. and A.6. in Appendix C are the first two-step problems that arise in the set.

A.7., A.8., & A.9. skills: Subtraction, addition, multiplication, division, and multiple steps

The word problems are still becoming more complex. The information within the problems is changing and is now offering the students more givens that they must identify and filter according to what they need to use to solve it, and the manner that they will use them in. A.8. and A.9. in Appendix D are the first three-step problems to appear, but have been gradually introduced within the course of problems.

A.10. & A.11. skills: Addition, subtraction, multiplication, division, and multiple steps

These last problems involve not only multiple-steps, but also multiple questions. Students must be able to identify not only the givens in the problems, but all of the questions that the problem is asking. If students have a method in which they organize the information that is provided in the problem, then they will be able to clearly see what the question is, and if there are multiple questions asked. Some confident students like to show their mathematical skill by finishing the problem quickly before their peers. On occasion this results in missed portions and skipped questions. A.10. and A.11. give the students the opportunity to witness problems that are multi-step and have multiple questions. The problems in Appendix E will also give them an example on what to strive for in the creation of their own generated problems.

Varying the Level

Since classrooms are not homogeneously grouped, problems must be altered to meet the needs of the individuals who do not reach the expectations to be considered on-grade level. The following examples are word problems intended for students who are below-grade level. The word problems have been altered by reducing the number of problems, smaller digits, and less steps. These students may also need the assistance of manipulatives to aid them in finding the ultimate solutions. Some manipulatives they can use to help could be base ten blocks and/or counters. Problems A.1. through A.11. are models of how to record all of the steps to solve a word problem and the same process can be applied to problems B.1. through B.6.

Problems B.1. through B.6. in Appendix F

Intended for students below-grade level

Just as problems are needed for students who are below-grade level, students who are above-grade level need to have their needs met as well. The following problems are designed to challenge them and involve higher numbers that require regrouping as well as problems that have multiple steps. Depending on the mathematical skills of the students, the numerical values in the problems given may be increased to suit the needs of the above-grade level students. The given problems are meant to be adapted for each individual classroom, so if the need presents itself, more complex and challenging numbers may be substituted.

Problems C.1. - C.10. in Appendix G

Intended for students above-grade level

Discussion

While the problems are being completed the teacher must lead the discussion that analyzes the problems in depth. The discussion should explore the structure of the problems and methods of understanding them in order to find a solution. The teacher should facilitate the conversation but allow for student explanations and reasoning. This important step will allow for immediate feedback and an opportunity for students to share their ideas and listen to their peers.

Guided Practice

Guided Practice is the interim step between teacher-directed instruction and independent practice. This is an important piece to the learning process, and unfortunately is sometimes skipped. Guided practice provides students with the time to practice the skill and understand the concept while getting prompted and immediate feedback from the teacher. The teacher is essentially guiding the student to the correct solutions, while giving them a degree of independences. This is also an opportunity for students to practice the skill without penalty to their grade. This can be done as whole class instruction on the overhead projector with student participation during the creation of problems. One problem may be created, solved, and justified by a multitude of students, but all under the watch of the entire class.

Independent Practice

During the Independent Practice portion of the lesson, the students will have received teacher-directed instruction and will have had opportunities to experiment with their own problems. The independent practice is where the students will have their chance to shine and to show how much they have learned and accomplished. Students will now begin to generate their own problems and thus, become the teacher.

To write their own word problems, students will choose a series of equations to include in their problems. They must first pick a basic problem (such as: 2 + 3 = 5), then create the inverse equation (such as: 5 - 3 = 2). The teacher must also demonstrate and make sure the students are aware of this principle with lager numbers. The reasoning behind this is so that the understanding that certain facts are connected to each other. Many students know 1 + 1 = 2, as well as 2 - 1 = 1, but the goal is for them to look at those problems as a whole and see how they are connected. They must be able to see that if they have 1 toy and then are given another toy, they then have 2 toys, but if one of those is taken away, they then have only 1 toy once again.

After the students create two basic equations that are connected, they can then write the first set of problems. They must use correct wording in their question that indicate which operation to utilize, as well as include the necessary information needed to solve the problem. They can then fill out the Step Chart (see example in Appendix H) to recheck their own problem and also to use as an answer key when other students solve.

Once students become fluent in converting basic one-step equations into word problems and explaining their actions, they will then be ready to create word problems with multiple steps. They must then take a one-step equation and make it multi-stepped. This may mean that a word problem requires the reader to use a multiplication equation and then add (2 x 3 = 6, 6 + 4 = 10), or a subtraction equation that then must be divided (10 - 4 = 6, 6 / 3 = 2). While students are doing this they must also remember that the inverse problem must be created so that they can continue to make the connection about their relationships.

Lesson Plan Timeline

Since the focal point of the unit is the student generation of word problems, there is an expectation that the students will have a somewhat firm grasp of basic facts and operations. Multiplication and division are not introduced until the second half of the year, so the unit should fall after those operations have been taught. It is also to be hoped that before the unit is started, that the students have some exposure to the idea of inverse relationships in mathematical operations. To allow enough time for students to achieve mastery of the inverse concept, it should be introduced in the beginning of the year and practiced (or discussed) until the unit begins.

Lesson #1

Exploration of word problems and modeling of discussion

The word problems are read together (teacher and students) but not with the intention to solve. They are read with the intention to explore and discuss the methods and strategies that could be used to solve. The teacher and the students can share their ideas about how to go about identifying which operations are necessary to arrive at the answer and any other issues that may arise.

Lesson #2

Introduction of the Step Chart and identifying the givens

The teacher introduces and models the use of the Step Chart in Appendix I to the students. Students are also encouraged to engage in conversations about how to identify the given information in the word problems, as well as comparing and contrasting the two inverse problems.

Lesson #3

Guided practice solving problems and discussion

Students are taken step by step through the solving of the word problems (several problems of each type or set should be considered before going on to the next step). Discussions about how problems are solved and the relationship between inverse operations are encouraged, especially those led by students.

Lesson #4

Composition of equations and inverse equations

After students are comfortable with the language of word problems and the steps in which to solve them, they will then take the step of writing the equations (and inverse equations). The equations that they compose will be used in their word problems. They must write them starting with a one-step equation with the basic operation of addition or subtraction. After that the equations must get progressively more difficult, utilizing multiplication and division, as well as multiple steps.

Lesson #5

Composition of word problems

Students will now integrate their original equations into carefully worded problems. Since they are the creators and in essence, the teachers of their own problems, they must write their problems in the Step Chart. The Step Chart will allow the students to think about their choice of words and digits, making sure the problems they compose are coherent and can be solved.

Lesson #6

Solving problems of peers

When the students are finished composing their own problems, they will be able to trade with their peers. Having another person look at and attempt to solve the problems will make the student authors accountable for what they compose. It is a good assessment tool for the students to discover if they have written a quality word problem that is challenging but also clear and concise.

Bibliography for Teachers

Bloom's Taxonomy. Learning Skills Program. 4 April 2004 http://www.coun.uvic.ca/learn/program/hndouts/bloom.html>. This is a quick reference guide to the steps in Bloom's Taxonomy. It can be used to evaluate what higher level thinking skills teachers are requiring their students to perform.

Carpenter, Thomas P., Elizabeth Fennema, Megan Loef Franke, Linda Levi, Susan B. Empson. *Children's Mathematics.* Portsmouth: Heinemann, 1999. This book addresses the problems that students have within the classroom relating to fully understanding mathematical operations. It gives strategies for teachers that can help foster student development in math.

Holt, John. *Instead of Education: Ways to Help People Do Things Better* . New York: E.P. Dutton & Co., Inc., 1976. This reference book discusses the need for student synthesis in the classroom and reasoning behind the necessity of it.

Lampert, Magdalene. *Teaching Problems and the Problems of Teaching.* New Haven: Yale University Press, 2001. A teacher's journey in teaching only by way of word problems is explained in this book. The author examines each student's work and every word spoken. It is useful to see how an analysis of student work and discussion can give insight on difficulties students are having or even new ways of explaining problems.

Ma, Liping. *Knowing and Teaching Elementary Mathematics.* Mahwah: Lawrence Erlbaum Associates, Publishers, 1999. Teachers in China have high expectations of their students and the students rise to the challenge. This book is a comparison between the teaching methods and practices of American and Chinese teachers. It is enlightening to see how those in China seem to perform "miracles" in the classroom, and helps define strategies that help them facilitate such success.

McTighe, Jay, and Steven Ferrara. *Assessing Learning in the Classroom.* Washington, DC: National Education Association, 1998. This short guide defines the different methods of assessment that a teacher can use in the classroom. Teachers can then work backwards by choosing what type of assessment to use to help define what they expect their students to learn and complete.

Piaget, Jean. *To Understand Is to Invent: The Future of Education.* New York: Grossman Publishers, 1973. Piaget believed that child development impacted the learning of certain mathematical skills. To be more exact he explains that some skills cannot be learned

at all until the child graduates to a higher phase because he/she is just not developmentally ready to understand a certain concept. His studies explain the necessity of student generated work and synthesis within the classroom.

Slavin, Robert E. *Educational Psychology: Theory and Practice* . Boston: Allyan and Bacon, 2000. This reference book defines Piaget's stages which can be useful in modifying mathematical lessons for the varying grade levels.

Student Resources

Students can use any text book they receive in class to help them evaluate and assess word problems. The goal is to fully understand word problems, so any word problem that is on their level is acceptable. As the facilitator for learning, the teacher can preview the text book and give specific guidelines as to which pages may be used and where to find helpful problems.

Burns, Marilyn. *I Hate Mathematics! Book* . Boston: Little Brown and Company, 1975. Burns is an author who knows what kids like and need to learn mathematical concepts. She incorporates games and puzzles in such a way that the students are busy being solution detectives and don't realize they are doing math!

Burns, Marilyn. *Math for Smarty Pants.* Boston: Little Brown and Company, 1982. This book may be useful for those students who are above grade level and finish everything before the rest of the class. It gives word problems a twist and makes them challenging for students.

Chapman, Carolyn. *Word Problems: 3rd Grade Math (Real-Life Math)* . Redding: Rainbow Bridge Publishing, 2002. This book includes problems that will relate to the lives of the students. This is practical math for students by making a concept that is hard to understand applicable to their own lives.

Materials for Classroom Use

Materials for whole class interaction and guided practice:

Overhead projector and markers

Step Chart with problems A.1 - A.11. on transparencies (with rest of chart left blank to fill in during the whole class interaction)

Scrap paper for students

Pencils

Manipulatives (counters, base ten blocks, etc.)

Materials for independent practice:

Blank Step Charts (See Appendix A)

Scrap paper

Pencils

Manipulatives (counters, base ten blocks, etc.)

Appendix A

A.1. Tony and I were comparing our baseball card collections. I have 5 less than Tony. I have 19. How many cards does Tony have?

A.2. Tony and I were comparing our baseball card collections. I have 5 less than Tony. Tony has 24. How many cards do I have?

First Step: Identify the givens and the question

A.1.

Givens:

- I have 5 cards less than Tony
- I have 19 cards

Question:

- How many cards does Tony have?

A.2.

Givens:

- I have 5 cards less than Tony
- Tony has 24 cards

Question:

- How many cards do I have?

Second Step: Problem Analysis

A.1.

Addition must be done in this problem. It states that I have 5 less than Tony, so Tony must have 5 more than me. Since I have 19 and Tony has 5 more, then addition should be used.

A.2.

The problem states that I have 5 cards less than Tony and that Tony has 24. If I have 5 less then 5 should be subtracted from 24 to find my total number of cards.

Third Step: Identify the operation(s) needed to find the answer and then solve

A.1.

Addition

19 + 5 = 24 cards

A.2.

Subtraction

24 - 5 = 19 cards

Fourth Step: Identify the connection between both problems and reasoning.

It first can be seen that the problems are connected because they utilize the same numbers within both equations. The equations use inverse operations to solve, addition and subtraction. It can be seen that when 5 cards are taken from 24 cards there are 19 cards left and when those 5 cards are added back to the 19 cards, then there are 24 cards once again. Simply translated, the problem shows that if I have 5 cards less than Tony, then Tony has 5 cards more than I do, so to find the number of Tony's cards I would add 5 to my cards. Thus the solution is 5 + 19.

Appendix B

A.3. Tony and I were comparing our baseball card collections. I have three times as many cards as Tony has. Tony has 9 cards. How many do I have?

A.4. Tony and I were comparing our baseball card collections. I have three times as many cards as Tony. I have 27 cards. How many cards does Tony have?

First Step: Identify the givens and the question

A.3. Givens:

- I have 3 times as many cards that Tony has
- Tony has 9 cards.

Question:

- How many cards do I have?

A.4. Givens:

- I have three times as many cards that Tony has.
- I have 27 cards

Question:

- How many cards does Tony have?

Second Step: Problem Analysis

A.3.

The problem clearly says that I have 3 times as many cards as Tony and Tony has 9. If I have 3 times the amount, then multiplication must be used to find my total number of cards.

A.4.

The problem states the number of cards that I have and that I have 3 times the amount of Tony. He must then have only a third of the amount of my cards. My amount of cards should be divided by 3 to find Tony's number of cards.

Third Step: Identify the operation(s) needed to find the answer and then solve

A.3.

Multiplication

3 x 9 = 27 cards

A.4.

Division

27 / 3 = 9 cards

Fourth Step: Identify the connection between both problems and reasoning.

The problems are connected because they utilize the same numbers within both equations. They are from the same "multiplication/division family." The equations use the inverse operations of multiplication and division to solve. It can be see that when there are 3 groups of 9 cards that there are 27 cards in all, and when those 27 cards are split back, or divided, into 3 groups there are 9 cards in each group.

The function that is expressed in the problem A.3. is multiplication (by three). A.4. gives the result of the multiplication fact and then asks for the multiple which requires performing the inverse operation of division.

Appendix C

A.5. Tony and I were comparing our baseball card collections. I have three times as many cards as Tony and 4 more besides. Tony has 9. How many do I have?

A.6. Tony and I were comparing our baseball card collections. I have three times as many cards as Tony and 4 more besides. I have 31. How many does Tony have?

First Step: Identify the givens and the question

A.5. Givens:

- I have 3 times the amount of cards that Tony has with 4 more besides
- Tony has 9 cards

Question:

- How many cards do I have?

A.6. Givens:

- I have 3 times the amount of cards that Tony has with 4 more besides

- I have 30 cards

Question:

- How many cards does Tony have?

Second Step: Problem Analysis

A.5.

The problem first states that I have 3 times the amount of cards as Tony and that Tony has 9 cards. To find the first step of the problem 9 must be multiplied by 3. I then not only have 3 times the amount of Tony, but also 4 more cards besides. Since I have 4 more, then 4 must be added to the product of 9 times 3 to find my number of cards.

A.6.

This problem states that I have 30 cards and that is 3 times the amount of Tony plus 4 more. If my total number of cards (30) is 3 times the amount and 4 more than Tony, then he must have 4 less than 30 and a third of that number of cards. So 3 must be subtracted from 30 and the difference must then be divided by 3 to find Tony's total number of cards.

Third Step: Identify the operation(s) needed to find the answer and then solve

A.5.

Multiplication and Addition

3 x 9 = 27

27 + 4 = 31 cards

A.6.

Subtraction and Division

31 - 4 = 27

27 / 3 = 9 cards

Fourth Step: Identify the connection between both problems and reasoning.

The problems are connected because they utilize the same numbers within both equations. The equations use the inverse operations of multiplication with addition and subtraction with division to solve. It can be see that when there are 3 groups of 9 cards that there are 27 cards plus 3 additional cards making 30 cards in all, and when those 30 cards have those 3 additional cards subtracted to make 27 cards which are then divided, into 3 groups there are 9 cards in each group. The operations are also undone in the opposite order (multiplication and addition inverted and opposite are then subtracted and divided).

Appendix D

A.7.a. Tony and I were comparing our baseball card collections. Tony gave me 6 cards and then we both had 10. How many cards did I have before he gave some to me?

A.7.b. How many cards did Tony start with?

A.8.a. Tony and I were comparing our baseball card collections. I had 4 cards and Tony had 16. Tony promised to give me some cards so we could have the same number of cards. How many did I end up with?

A.8.b. How many did Tony give me?

A.9. Tony and I were comparing our baseball card collections. Tony gave me 6 cards and then we had the same number. I started with 4. How many cards did Tony start with?

First Step: Identify the givens and the question

A.7.

Givens:

- Tony gave me 6 cards
- We both have 10 after he gave me the cards

Question:

- How many cards did I have before he gave some to me?
- How many cards did Tony start with?

A.8.

Givens:

- I had 4 cards

- Tony had 16 cards
- Tony gave me cards so we both had the same number

Question:

- How many cards did I end up with?
- How many did Tony give me?

A.9.

Givens:

- Tony gave me 6 cards
- I had 4 cards to start
- We both had the same number of cards after he gave me 6

Question:

- How many cards did Tony start with?

Second Step: Problem Analysis

A.7.a. Tony gave me 6 cards and then I had 10. I had 6 less cards before he gave any to me. If I subtract 6 from 10 the amount of cards that I started with is found.

A.7.b. Tony gave me 6 cards and then we both had 10. He had 6 more cards before he gave any away. The 6 cards must be added to the 10 he has in the end to find the total number of cards that Tony had before he gave any away.

A.8.a. In this problem we know that we both end up with the same amount of cards. To find that sum total all of the cards should be combined, adding 4 to 16. Then since there are two of us who are splitting the cards, the sum should be divided into 2 groups.

A.8.b. If we find out that both parties end up with 10 cards after Tony gives some away, then we know that Tony had 10 in the end. He started with 16 and 10 should be subtracted from that amount to find the number that he gave away.

A.9. We know that Tony gave me 6 cards to add to my 4 cards. After his donation I have 10 cards and the problem states that we then both own the same amount of cards. 10 should be multiplied by 2 since there are two groups of 10 cards. The product of 20 is our combined number of cards. Since I had four to begin with and we are looking for Tony's original number, we can subtract my original four from the collective total of 20 cards.

Third Step: Identify the operation(s) needed to find the answer and then solve

A.7.a.

Subtraction

10 - 6 = 4 cards

A.7.b.

Addition

10 + 6 = 16 cards

A.8.a.

Addition, division.

4 + 16 = 20 cards (altogether)

20 / 2 = 10 cards

A.8.b.

Addition, division, subtraction

4 + 16 = 20 cards (altogether)

20 / 2 = 10 cards

10 - 4 = 6 cards

A.9.

Addition, multiplication, subtraction

4 + 6 = 10 cards (each)

10 x 2 = 20 cards (altogether)

20 - 4 = 16 cards

Fourth Step: Identify the connection between both problems and reasoning.

All of the problems are about the same situation, but they give different pieces of information about that situation. We must reconstruct other information using the relationships given in the problem. In A.7. the student must notice that “I” will have 10 after Tony gives “me” 6, so to find “my” original total, 6 must be subtracted from 10. In A.8. it must be observed that both persons will have the same number of cards after Tony gives some away. To find how many both have after the trade, the total of all the cards must be found, 4 + 16 = 20 cards altogether. Then by dividing by two (the number of people the cards are split among) the number of cards that each person has can be found, 20 / 2 = 10 cards each. Finally, by taking the number of cards “I” have at the end, which is 10, and subtracting what “I” had to start, which is 4, it can be found how much Tony gave to “me,” which is 6 cards. Finally, by using the inverse operations used in A.8., A.9. can be solved. The first step to A.9. is adding the two numbers together, because Tony gave “me” 6 cards in addition to the original 4. That gives “me” 10 cards total (the teacher must be very explicit about the observation that the total number of cards remains the same). The problem also stated that both people had the same amount of cards after Tony gave some away, so if 10 is multiplied by 2 (the number of people that each have 10 cards), it can be found that there are 20 cards altogether. Finally, to find out how many Tony had to start, the number of cards that “I” had (4) can be subtracted from the total number of cards between both people (20) to equal the 16 cards that Tony had to start.

Appendix E

A.10.a. Tony and I were comparing our baseball card collections. I had twice as many cards as Tony and 3 more besides. I have 21. How many did Tony have?

A.10.b . Our friend Mark came and wanted to share our cards but didn’t have any of his own. How many cards do Tony and I have to begin with? If we share our cards equally, how many will each person get?

A.11.a . Tony and I were comparing our baseball card collections. I had twice as many cards as Tony and 3 more besides. Tony has 9 cards. How many cards do I have?

A.11.b. Our friend Mark came and wanted to share our cards but didn’t have any of his own. We split our cards equally. How many cards do the three of us have altogether?

First Step: Identify the givens and the question

A.10.

Givens:

- (a) I had twice as many cards as Tony and 3 more besides
- (a) I had 21 cards
- (b) Mark wants to share our cards and we will all have an equal number of cards

Question:

- (a) How many did Tony have to begin?
- (b) How many cards do Tony and I have to begin with?
- (b) How many can we each get (Tony, Mark, and I) to have an equal number of cards?

A.11.

Givens:

- (a) I had twice as many cards as Tony and 3 more besides
- (a) Tony has 9 cards
- (b) Mark is sharing our cards and between the three of us, we each get 10 cards

Question:

- (a) How many cards do I have?
- (b) How many cards do the three of us have altogether?

Second Step: Problem Analysis

A.10.a. This problem resembles A.5. and A.6. It states that I have 21 cards, which is twice the amount plus 3 more than Tony. He must then have three less than 21 and then half of that difference. Subtraction and division must be used to find Tony's total number of cards.

A.10.b. The problem then asks how many cards do Tony and I have combined. By taking the amount I have (21) and Tony's found amount of (9) and adding them, the sum of our cards may be found. Mark then enters the problem with no cards of his own and wants to share. There are then three groups that must divide the 30 cards up evenly.

A.11.a. Tony has 9 cards. I have twice as many plus 3. If I have twice as many (plus 3) then the amount of cards that Tony has (9) should be multiplied by two. Then the 3 additional cards should be added to find my sum total of cards.

A.11.b. Mark comes and wants to share the cards. We know that combined we have 30 cards (Tony's 9, plus my 21). Even if Mark comes and we split our cards 3 ways, the sum total of cards will not change and we will still have 30 cards altogether. Multiplication can be used if it is found that each person gets 10 cards and if there are 3 groups, the total number of cards can be found.

Third Step: Identify the operation(s) needed to find the answer and then solve

A.10.

(a) Subtraction, Division

21 - 3 = 18

18 / 2 = 9 cards

(b) Addition, Division

21 + 9 = 30

30 / 3 = 10 cards each

A.11.

(a) Multiplication, Addition

9 x 2 = 18

18 + 3 = 21 cards

(b) Multiplication

10 x 3 = 30 cards altogether

Fourth Step: Identify the connection between both problems and reasoning.

The same numbers are used in both problems. The questions change and therefore require reverse operations to be used. When A.10.a. is compared with A.11.a., it can be seen that subtraction and division are used, to find Tony's total number of cards and then multiplication and addition are used to find "my" total number of cards. The key words are *twice* , and *more besides* . After the given information is found, those key words tell which operations are necessary. In A.10.b., a third person is added to the mix, so the total number of cards to be shared among the three must be found before finding out how many can be dispersed evenly. Since the problem asks how many can each person have, it must be divided. In A.11.b. the opposite is asked. The total number of cards that each person has is given, and the question then asks how many the group has altogether. Since there are three groups of cards and the total number of cards among all three is asked for, then multiplication is the operation used.

It can also be noted that these problems parallel A.5. and A.6. but are more complex, showing how scaffolding is important for these word problems.

Appendix F

B.1. Tony and I were comparing our baseball card collections. I have 1 less card than Tony. I have 4 cards. How many cards does Tony have?

Operation utilized: addition

B.2. Tony and I were comparing our baseball card collections. I have 1 less card than Tony. Tony has 5 cards. How many cards does Tony have?

Operation utilized: subtraction

B.3. Tony and I were comparing our baseball card collections. I have twice as many cards as Tony. Tony has 5 cards. How many cards do I have?

Operation utilized: multiplication (or repeat addition)

B.4. Tony and I were comparing our baseball card collections. I have twice as many cards as Tony. I have 10 cards. How many cards does Tony have?

Operation utilized: division

B.5. Tony and I were comparing our baseball card collections. I have twice as many cards as Tony and 2 more besides. Tony has 5 cards. How many cards do I have?

Operations utilized: multiplication (or repeat addition) and addition

B.6. Tony and I were comparing our baseball card collections. I have twice as many cards as Tony and 2 more besides. I have 12 cards. How many cards does Tony have?

Operations utilized: subtraction and division

Appendix G

C.1. Tony and I were comparing our baseball card collections. I have 19 less than Tony. I have 33. How many cards does Tony have?

Operation utilized: addition with regrouping

C.2. Tony and I were comparing our baseball card collections. I have 19 less than Tony. Tony has 52 cards. How many cards do I have?

Operation utilized: subtraction with regrouping

C.3. Tony and I were comparing our baseball card collections. I have 9 times as many cards as Tony. Tony has

12 cards. How many do I have?

Operation utilized: multiplication

C.4. Tony and I were comparing our baseball card collections. I have 9 times as many cards as Tony. I have 108 cards. How many cards does Tony have?

Operation utilized: division

C.5. Tony and I were comparing our baseball card collections. I have 9 times as many cards as Tony and 9 more besides. Tony has 9. How many cards do I have?

Operations utilized: multiplication and addition with regrouping

C.6. Tony and I were comparing our baseball card collections. I have 9 times as many cards as Tony and 9 more besides. I have 90 cards. How many cards does Tony have?

Operations utilized: subtraction with regrouping and division

C.7. Tony and I were comparing our baseball card collections. Tony gave me 6 cards and then we both had 23. How many cards did I have before he gave some to me? How many cards did Tony start with?

Operations utilized: subtraction with regrouping, multiple steps and questions

C.8. Tony and I were comparing our baseball card collections. I had 17 cards and Tony had 29. Tony promised to give me some cards so we could have the same number of cards. How many did Tony give me? How many did I end up with?

Operations utilized: addition and subtraction with regrouping, multiple steps and questions

C.9. Tony and I were comparing our baseball card collections. Tony gave me 6 cards and then we had the same number. I started with 17 cards. How many cards did Tony start with?

Operations utilized: addition and subtraction with regrouping, multiple steps and questions

C.10.a. Tony and I were comparing our baseball card collections. I had 5 times as many cards as Tony and 6 more besides. I have 41 cards. How many did Tony have?

C.10.b. Three of our friends (Pat, Jorge, and Mike) came and wanted to share our cards. Pat had 6 cards, Jorge had 8 cards and Mike had 4 less than Tony. If we share all our cards equally how many will each person get?

Operations utilized: division, addition and subtraction with regrouping, multiple steps and questions

C.11.a. Tony and I were comparing our baseball card collections. I had 5 times as many cards as Tony and 6 more besides. Tony has 7 cards. How many cards do I have?

C.11.b. Three of our friends (Pat, Jorge, and Mike) came and wanted to share our cards. Pat had 2 less than Jorge, Jorge had 1 more than Tony, and Mike had 4 less than Tony. If we share all of our cards equally how many will each person get?

Operations utilized: multiplication, addition and subtraction with regrouping, multiple steps and questions

Appendix H

STEP CHART (example)

A: Word problem

Madeline and Pat were comparing their doll collection. Madeline has 3 less than Pat. Madeline has 2 dolls. How many dolls does Pat have?

B: Inverse Word problem

Madeline and Pat were comparing their doll collection. Madeline has 3 less than Pat. Pat has 5 dolls. How many dolls does Madeline have?

First Step: Identify the givens and the question

A:

Givens:

- *Madeline has 3 less than Pat*
- *Madeline has 2 dolls*

Question:

- *How many dolls does Pat have?*

B:

Givens:

- *Madeline has 3 less than Pat*
- *Pat has 5 dolls*

Question:

- *How many dolls does Madeline have*

Second Step: Identify the operation(s) needed to find the answer and then solve

A:

Operation identified

Addition

Equation written and solved

2 + 3 = 5 dolls

Pat has 5 dolls.

B:

Operation identified

Subtraction

Equation written and solved

5 - 3 = 2

Madeline has 2 dolls.

Third Step: Identify the connection between both problems and reasoning.

The connection between the problems is explained.

Addition is the opposite of subtraction. If 2 + 3 = 5, then 5 - 2 must equal 3, and 5 - 3 must equal 2.

The reasoning behind the method and solution is given.

A: It is stated that Madeline has 2 dolls but also has 3 less than Pat. If Madeline has 3 less, then Pat then must have 3 more than Madeline. Since Pat has 3 more than Madeline, 3 must be added to the 2 dolls Madeline has to come up with Pat's total of 5.

B: It is stated that Pat has 5 dolls and that Madeline has 3 less than Pat. Since Madeline has 3 less than the given number of 5 that Pat has, 3 must be subtracted from 5 to total Madeline's 2 dolls.

Appendix I

Blank Step Chart for student use

Word problem: Inverse problem:

First Step: Identify the givens and the question

Givens: Givens:

Question: Question:

Second Step: Problem Analysis

Third Step: Identify the operation(s) needed to find the answer and then solve

Fourth Step: Identify the connection between both problems and reasoning.

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