Curriculum Units by Fellows of the Yale-New Haven Teachers Institute
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Using Algebra Word Problems to Explore Problem Space

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by Anthony B. Wight

Introduction

Words and their meanings may present the greatest obstacle to mathematical problem solving for students. Reading for understanding and writing for explanation are at the foundation of successful learning in all disciplines. It is quite clear that as a high school teacher, I depend greatly on my colleagues teaching in younger grades to help students learn to read and write. Every year I rediscover how much I, too, need to coach and encourage basic communication skills in my students. If any data drives this point dramatically, it is the recent "Reading at Risk" survey which found that "fewer than half of all Americans over 18 read novels, short stories, plays or poetry . . . and the pace at which the nation is losing readers, especially young readers, is quickening."1

In other words, I cannot just "teach math" with the assumption that all can speak, read and write with ease any more than I can assume that all my matriculating ninth graders can solve problems using fractions, decimals, percents, measurement and graphing. Our students vary greatly in all aspects of their development and understanding. Figuring out where each individual stands along numerous continua of conceptual and expressive development ought to be a shared task of teachers of all grades working together to develop and repeatedly teach useful "tools" for analyzing, thinking about and solving problems (and not only mathematical ones).

Word problems represent the "real world" of money, measurement, information, consequences and arguments. At all levels, mathematics curricula should include and embrace word problem solving. Such exercises not only review and reinforce computational skills, but also they challenge students to shift from verbal situations to patterns, tables, graphs, equations and back. Building problem sets, which explore key arithmetic concepts and relate symbols to the concrete realities of students' lives, can moderate the abstract and remote nature of "algebra" for many students.

In this unit, I look at a few types of algebra problems and discuss ways of solving them that might work on a range of grade levels. First, I briefly discuss the Vertical Team2 concept and its value for mathematical problem solving. Second, I introduce the idea of "problem space", a phrase frequently used by our seminar leader, Professor Howe. Third, I propose that along with our curricular sequencing we look for and practice deliberate techniques that work each year for progressively more complex and intricate problems. Students thus will be building their own “toolkits” for algebraic problem solving skills, which they can expect to master and apply as they advance through the grades. Finally, I present and discuss several model problem sets designed for classroom activities in developing algebraic problem solving.

The Case for Mathematical Vertical Teaming

The teaching of mathematics requires critically examining the ways in which students understand basic concepts and build upon these to formulate new ideas. The New Haven Public Schools’ Mathematics Department has initiated a collaborative endeavor called “vertical teaming” to bring together math teachers across grades 5-12 in clusters roughly consisting of pathway schools for students. Vertical TeamTM is a trademarked term for a concept developed by The College Board to bring together teachers from different grade levels in a discipline to work cooperatively in developing and implementing a vertically aligned program aimed at helping students acquire the academic skills necessary for success in the Advanced Placement Program.3 On the one hand, it may appear to be a clever marketing strategy to assure future participants in AP courses and (paying) customers for The College Board’s AP exams--and it is. On the other hand, who among us does not want to have our schools “build rigorous curricula; promote access to AP for all students; introduce skills, concepts, and assessment methods to prepare students for success when they take AP; and strengthen curriculum and increase academic challenge for all students”?4

Some New Haven teachers have begun talking to each other about their goals and frustrations in teaching math with the aim of developing a consistent set of expectations for student accomplishment at each grade level. Knowing what is expected in the grades-before and grades-following the one(s) you are teaching will hopefully lead to better opportunities for students to progress in mastery of skills and in complexity of problem solving. Numeracy tasks introduced in elementary classes need reinforcement right through grade 12. By working together, vertical teams hope to stop the blaming of prior teachers for failing to adequately teach fundamentals (i.e., fractions, decimals, percent, graphing, vocabulary, technology) and recognize our shared responsibility for helping all students attain their greatest success along a continuum of learning.

So, in a nutshell, the idea is to “improve academic performance for all students in earlier grades by introducing skills and concepts needed for success in AP and other challenging courses.” 5

Vertical teaming is a “win-win” situation. Isn’t this what all schools have been striving to do all along with curricular standards, curriculum outlines and grade level goals in each discipline? Yet, how often have teachers from all grade levels in a discipline been encouraged and enabled (given the time and yes, even the money) to meet together regularly over a long period of time to really look at and talk about what they and their students are doing? In our seminar, Professor Howe floated the idea that perhaps schools should be “teacher-centered” instead of “student-centered”. After all, who knows best what your next year’s incoming class can and cannot do in arithmetic, algebra and problem solving? Teacher-driven reform lifts up the hope of real staying power in education, because teachers know that reform takes constant tending and checking, adjusting and praising. This seminar has been one form of vertical teaming--we listened to each otherr’s struggles with our teaching at different levels, tried to solve problems together and began to form ideas of what, when and how to unfold mathematical understandings with our students.

A book that should be read in developing a critical foundation for the teaching of word problems is Magdalene

Lampert's *Teaching Problems and the Problems of Teaching* (Yale University Press, 2001). In this book-length reflection on the teaching of fifth grade math by a university professor, what stands out is her patient questioning and listening processes while working with students along with her deep reflections afterward in preparation for the next day's class. Lampert had the extraordinary opportunity of doing her research on teaching and thinking by instructing an elementary class one hour daily for a year, along with her college teaching responsibilities. Her book is itself a strong argument for a "vertical" view of mathematical learning. She brings to her fifth grade class both an unclouded focus on the importance of the work each individual is doing and clear sense of how that work is connected to the lifelong development of math's "powerful ideas". Of course, as teachers in demanding urban schools, my seminar colleagues and I marveled at the depth of reflection and the sheer time Lampert could afford to try to grasp the intricacies of each fifth grader's struggle toward understanding.

The Concept of "Problem Space"

One of the deadly sins of teaching math to which we all feel tempted at times by desperation, exhaustion or fear of approaching standardized tests is trying to give students disparate tricks or shortcuts to get answers to problems rather than engaging them in explorations and self-generated thought processes which in the long run would build better numeric and algebraic understanding. Often my students can call out an acronym ("FOIL"), phrase ("Please Excuse My Dear Aunt Sally") or other "handy" mnemonic device that they recall from a prior math class (perhaps even from my class), but they have no idea what to do with it or no understanding of the core math ideas, which underlie it.

The "Craft of Word Problems" seminar has given me an opportunity to engage with math colleagues from several teaching levels to tease out the ways in which the language and structure we use enhances or inhibits the growth of mathematical reasoning and understanding. I teach math full time to students in grades 9-12 with a very broad range of academic ability and intellectual curiosity. The seminar has enabling me to select and design word problems more tailored to the needs of my students. In my teaching, one of the greatest challenges is getting students to read word problems for broad understanding without getting lost in details and abandoning hope. And it is here that the idea of exploring the problem space is most helpful. By crafting sets of related word problems which build around a central math idea, students can be engaged in a thought process of discovering, comparing and contrasting, refining understanding, seeing questions from several angles of view . . .NOT just crunching numbers through an abstract formula to get the answer. "The hope is that exploring the problem space can give students a better rounded view of the issues involved in solving any one of the problems. Each problem by itself may be a challenge, but solving all the problems may make the solution of each one easier and more transparent, and may create a more lasting impression than any random individual problem."6 Word problems, well composed, ought to draw students into more subtle examination of the features of the "space" or mathematical habitat or territory where the problems reside.

Let me illustrate this point with a personal commentary:

The meaning of mathematics arises within authentic learning experiences and real life problem solving at all ages. When picking up my 4 year old grandson, Max, at his preschool one day last fall, he explained to me that, "There were 3 girls and 2 boys in our (pre) school program, but one girl left before you came to pick me up, so there are now only 2 girls and 2 boys, this many in all" (holding up 4 fingers). This is one of his real life settings of great value--he pays close attention to “who”, “what” and “how many.” As we pulled out of the parking lot to go home, he said to me, “Now there are only 2 girls and 1 boy, this many” (3 fingers). We tried to guess who would leave next and what that would do to the group remaining. (Our ensuing conversation illustrates what I mean by “exploring problem space”: a collection of related problems created around a theme to build student understanding of a mathematical idea.) I posed more questions to Max as we drove home: “If one more student leaves, how many students will be at school? How many students would have to leave to make the number of boys the same as the number of girls? How many students in all would then still be at school? Is there another possible answer that is true? How many students would be left then?” -- In this simple case, we were exploring grouping, subgroups, subtraction, the idea of equal numbers and whether “none” (zero) is a number in his 4-year-old mind. Max used his fingers as a simple calculator and “graphic organizer”. He must have had a good nap at school that day, because he stayed with the line of questioning most of the way home. Most days he falls asleep almost immediately in his car seat!

There seems to be a lot of math in social relationships at the pre school level and it is not much different in my high school, though the situations become a bit more complex! The words, numbers, and concepts used to address their lives are important for my students to understand if they are to achieve a sense of confidence and empowerment.

One of the most encouraging outlooks on teaching mathematics that we read and discussed frequently in our seminar is that of Thomas Carpenter, Megan Franke and Linda Levi in their book *Thinking Mathematically* (Heinemann Press, 2003): “…children throughout the elementary grades are capable of learning powerful unifying ideas of mathematics that are the foundation of both arithmetic and algebra.”7

All teachers of mathematics might benefit if the following statement were made into a poster and placed on classroom walls as a daily reminder of how active, interdisciplinary and authentic an experience the study of mathematics should be.

> Learning mathematics involves learning ways of thinking. It involves learning powerful mathematical ideas rather than a collection of disconnected procedures for carrying out calculations. But it also entails learning how to generate those ideas, how to express them using words and symbols, and how to justify to oneself and to others that those ideas are true.8

Thinking Mathematically is a book that should be read by teachers of mathematics at every grade level. It does not provide a magic solution to the difficulties students and teachers experience in learning and teaching math, but it does set a framework for developing “types of problems and forms of questioning that [the authors] have found useful for eliciting children’s thinking and fostering growth in mathematical understanding.” 9

Analytical Tools for Mathematical Fools

I doubt very few if any of our students and probably few of you reading this would want to be thought of as “fools.” Though, of course, many--if not all of us--engage in foolish behavior from time to time (perhaps even at this very moment). I donon’t in any way mean the term as a repudiation or judgment, but something much more grand. To act the fool is not (or at least not only) to be a clown, but to enthusiastically follow your heart (“a fool for love”), your greatest passion (“a dancing fool; a Huskies basketball fool”), your deepest curiosity (“a questioning fool”). To act the fool in this sense is the opposite of “conventional” thinking that suggests mathematics is dull, dry, difficult and deadening. If math is to be an adventure that lures students to *explore* and *engage* with big ideas, it will depend greatly on how we, as teachers, play it.

Teachers of math have developed numerous ways to attack problem solving: illustrating, graphic organizers, strategies, and techniques, etc. We all have our favorites. Vertical teams might spend lots of productive time sharing and demonstrating methodologies and developing those, which could become familiar implements in a problem solvers’ “toolkit”. The advantage of “vertical” consideration of any strategy would be building in students both a sense of what the strategy or technique can be used for and in helping students gain confidence and sophistication for use of the strategy. (My four year old grandson knows what a hammer is and loves to use one, but when he is in high school if he is still holding and swinging a hammer the way he does now, I will really have failed him!)

Some of my own favorite “analytical tools” will be used in discussing the word problem sets that follow. I try to suggest as many different strategies as I can to students, often challenging them to explain or do problems several ways to help them see what works best for them. Among the familiar “logical” hobby horses I have been known to ride are: Venn Diagrams, Charting, Eliminating Possibilities, Trial and Error, Working Backwards, Vee Charting, Unit Conversion and Unitary Rates. The most useful tool for my algebra students is a “5-Part Problem Solving Page”10 which allows students to use a variety of “learning styles” simultaneously in working through a problem with:

- Verbal Description
- Pattern Sketch
- Table of values
- Graphical Representation
- Equations

A ”multiple solutions” blank template for use with problems is included as Figure 2 in the appendix of this unit and use of the template is included in discussion of problem sets below.

All the various problem attack tools are useful, but mindful of what I said above about teaching procedures for just “getting answers” instead of for relational thinking and generalized understanding of the big ideas in math, it is possible to ‘lose sight of the forest (the big idea) while looking closely at the trees (a clever technique)’. A tool is only as good as the hand that holds it and the mind that knows when, where and how to apply it. (One of my favorite tools is a pair of pliers, but it makes a lousy hammer.) With this note of caution, I recommend highly another book series, which we examined in our seminar.

A “Singapore Sling”

The Curriculum Planning & Development Division of the Ministry of Education in Singapore has produced a series of Primary Mathematics Texts, which we found useful for study in our Institute seminar. We read the U.S. Edition of *Primary Mathematics, 6A and 6B* (Times Media Private Limited, 2003). The Singapore curriculum series offers insight into how one Asian school district is presenting students with learning materials. Considering the comparative high success of Asian and Asian-American students in mathematics, it is instructive to read through these materials. Several immediate impressions are worth noting here:

1. The series is comprised of slim paperback texts and linked workbooks (copies of which we did not have on hand). The texts are two color printed, uncluttered (compared to U.S. textbooks) and filled with clear graphics and almost spare verbiage.
2. The main feature of the package is constant use of the self described “Concrete->Pictorial->Abstract” approach to problem solving. “Students are provided with the necessary learning experiences beginning with the concrete and pictorial stages, followed by the abstract stage to enable them to learn mathematics meaningfully. This package encourages active thinking processes, communication of mathematical ideas and problem solving.”11
3. The very first unit in 6A is “Algebra” (no shilly-shallying here!). Following units are “Solid Figures” (drawing and nets), “Ratio” (ratio and fractions, ratio and proportion, changing ratios), “Percentage”, and “Speed.” In 6B, the units are “Fractions”, “Circles”, “Graphs”, “”Volume”, Triangles and 4-sided Figures, and “More Challenging Word Problems.”

Overall, the texts seem to send a message that the study of mathematics can be both serious and quite engaging. The format does encourage students to think about problems, and the constant use of clear graphics prods students to think in both organized and visual-artistic ways. (In a very unscientific sampling, I tried some of the exercises with my younger high school students who usually struggle with text materials and found that they generally made a good effort, were not intimidated or belittled by the problem presentations, and, by not getting frustrated, were able to talk about solutions they were trying.)

Carpenter’s thesis in *Thinking Mathematically* stresses that “it is important for all children to learn that arithmetic and algebra make sense and that arithmetic and algebra are grounded in a basic collection of big ideas…and there is a small list of fundamental numeric properties that account for all symbol manipulation in arithmetic and algebra."12

Reading Carpenter's book while reviewing the Singapore curriculum for sixth grade provided an interesting pairing of thoughtful pedagogy with a specific example of textbooks grounded in the same outlook. Chapter by chapter in Primary Mathematics formulaic " *rules* " (with accompanying sample calculations so familiar in U.S. texts) are almost nonexistent. Instead, the text is presented more in the form of a Socratic dialogue with students, designed to lead them into investigation and exploration of mathematical and algebraic *properties* . For too many students (and adults), "arithmetic represents a collection of unrelated and arbitrary manipulations of numbers and symbols, and algebra is perceived as a separate collection of meaningless procedures that are only tangentially related to arithmetic."13

Thinking Mathematically and the Singapore Curriculum should be of great interest to math vertical teams--both emphasize the importance of generative learning (that which "serves as a basis for acquiring new knowledge"14). And both have the goal of students learning arithmetic in ways that naturally build a foundation of understanding to support the learning of algebra.

Word Problem Sets

This unit develops a "multiple-meaning" approach to problems in an attempt to draw students in from any of several directions as they solve problems and explore mathematical ideas Since I teach primarily Algebra and Geometry, I developed problems and models that work within these subjects but which also are accessible for my students with less than grade level achievement. Thus, I hope my unit will also be useful for teachers and vertical teams seeking ways to build greater meaning in the progressive study of mathematics. For each set of problems I provide a brief overview discussion of the particular problem space and a sample or two of a complete solution to a problem.

1. Piling Up Blocks--Multiple Solutionss

During our seminar, my initial set of problems (see "Functions of *n* " below) designed to demonstrate the use of multiple solutions (the "5-way problem page") was too difficult for teachers in elementary grades to apply in their classes. With input from second and third grade teaching colleagues, I created a set of algebraic word problems, which could be solved by elementary students in at least one or more of the corners of the multiple solutions template. Algebra and geometry students should be able to solve these in all four of the template corners and in addition, if given one of the template corners completed, should be able to write a corresponding verbal description. It is not necessary to require that students "fill the page" with all four solution methods, but a great deal of inter-related learning can be discovered by having small teams of students take on the task of completing the page. Some students are very adept at pattern visualization, others at table building, while others may leap to define variables and set up equations. All middle and high school students should practice the skills of going from table to graph and from graph to table. Thus, while I list this problem set as a series of word problems (verbal descriptions), each question could be presented as a graph, table, equation or pattern sketch.

Including two different types of block/step patterns enlarges the problem space--relationships between "numbers of blocks" and "number of rows high". Each pattern has its own block-to-row relationship, but a comparison of the two relationships may yield more insights for students. (Two lines on the graph, two sets of equations, etc. are useful analytical tools for algebra students). The final problems ask students to generalize a formula (equation) for both of the patterns and then to make a comparison of the two patterns' results for a project involving 200 blocks. Given enough blocks or LegosTM, any student could construct these steps, but even younger students may see the value in solving mathematically when block counts get quite large. (Problem 1 below is solved for demonstration purposes on the template in Figure 1 of the appendix.)

1. Amir is arranging blocks. He puts one in the top row, 2 in the 2nd, 3 in the 3rd, etc. How many total blocks does he need to build 5 rows?
2. Amir is using blocks to build steps. He has a total of 55 blocks. If each block sits on two others, how many rows high will his steps be?
3. Amir wants to build block steps 10 rows high. How many blocks does he need if each block rests on two others?
4. Michelle decides to build steps similar to Amir's. She starts with a row 10 blocks long, how many blocks will she need if she wants her last row to have just one block?
5. Sara wants to build steps that go up one side and down the other. She decides to put each block directly on top of another. How many rows high will her steps be if she uses 10 blocks?
6. Sara wants to build block steps 5 rows high with just one block on the top. She will put each block directly on top of another. How many blocks does she need in all?
7. Alfred borrows Amir's 55 blocks and builds steps using Sara's pattern. How many rows high will his steps be? (Will he have any extra blocks?)
8. How many total blocks will Amir need to build n rows with his step pattern?
9. How many total blocks will Sara need to build n rows with her pattern?
10. If Amir and Sara each have 200 blocks to work with, what is the ratio of the number of rows that Amir will complete to the number Sara will complete?

2. "Go East Young Mathematicians"

In our seminar we discussed the generally poor performance of American students on international mathematics tests. Along with reading several books by educators who place great emphasis on development of algebraic understanding from an early age (see bibliography for suggestions), I looked at curricular materials from Singapore and Japan--two countries with track records of high success among their students on international math exams. The following two sections provide more information about these curricula and illustrative problem sets.

A. Singapore Approach: Concrete->Pictorial->Abstract

The problem solving approach in the Singapore Curriculum involves not so much a “strategy” as a graphical way of relating numbers to solve problems visually and logically. Algebraically, the level of abstraction (in the sixth grade texts studied) does not include emphasis on naming variables and writing equations. Since my ninth and tenth grade students at Cross Annex often have great gaps in their educational development, this approach may have real appeal as well as value in building up some of the missing foundation of numeric understanding. A pair of problems inspired by the Singapore series will illustrate the approach.

Sample #1: Jose had twice as much money as Ted. But after Jose spent $50 and Ted spent $20, they each had the same amount of money. How much money did each have at first?

Discussion: No formal equations are used in this problem solving approach. The information in the word problem is blocked out in units or bars to represent the money Ted and Joe each had at the beginning of the problem and after their spending. The key idea is to encourage students to look at the relationships between the boys’ amounts of money both before and after their spending. Solving requires identifying in the diagram (see Figure 3 in the appendix) the value of “1 unit” of money as just the difference between Jose’s spending and Ted’s spending (50 - 20 = 30) as shown in Figure 2. Ted started with one “unit” or $30 and Jose started with 2 “units” or $60.

Sample #2: Sam earned $140 doing yard work after school. His sister Kim earned $60 babysitting. When their grandmother gave them each an equal amount of money for new school clothes, Sam had twice as much money as Kim. How much money did their grandmother give each of them?

Discussion: using a “working backward” strategy and the graphic representation of the two students’ money AFTER the grandmother’s gift and BEFORE the gift can solve this problem. Again, the key is in looking at the relationship between the amounts of money the two students had at those times. Careful attention to the information in the problem is a must--as with all word problems. Visually representing the money amounts may keep students focused on the details. Although, students could define variables and solve this problem with equations, in this Singapore approach, equations are not necessary. Understanding what is going on with the numbers is the goal. Figure 4 in the appendix illustrates the solution to this problem using unit bars and shows that $140 - &60 -$60 = $20, the amount of the grandmotherr’s gift.

1. Jose had twice as much money as Ted. But after Jose sent $50 and Ted spent $20, they each had the same amount of money. How much money did each have at first?
2. Sam earned $140 doing yard work after school. His sister Kim earned $60 babysitting. When their grandmother gave them each an equal amount of money for new school clothes, Sam had twice as much money as Kim. How much money did their grandmother give each of them?
3. Matt and Nigel each had the same amount of money. Nigel spent $18 each week on lunch and Matt spent $24 on bus fares and lunch. When all of Matt’s money was spent, Nigel still had $120. How much money did each of the boys start with?
4. Mike and Tom were trading baseball cards. Mike had 40 more cards than Tom. After Tom gave Mike 12 cards, Mike had twice as many cards as Tom. How many cards did they have altogether?
5. Javon and Phyllis compare their music collections. Javon has three times as many CD’s as Phyllis and six more besides. If Phyllis has 14 CD’s, how many does Javon have?
6. Sheila had three times as much money as Donaya. After Donaya made $6 and Sheila spent

$17, they each had the same amount of money. How much did each have at first?
7. Ashlee had $35 and Jackie had $80. After they each earned an equal amount of money at their jobs, Jackie had twice as much money as Ashlee. How much money did each earn?
8. Mimi had $55 and Jackie had $80. How much money must Jackie give Mimi so the girls have the same amount of money?
9. Jackie had $80 and Mimi has $55. The girls went to the mall and each spent the same amount of money. On the way home they compare their money and find that Jackie now has twice as much as Mimi. How much did each girl spend?
10. Tony had $45 and Jon had $75. They go into business selling hotdogs agreeing to share all costs and profits equally. At the end of the first day, Jon has twice as much money as Tony. How much did they each make? How much does each now have?

The set above provides an opportunity to tackle word problems with relatively small amounts of data, but with some interesting relationships among the data. The Singapore graphics technique will work well on all of these problems, but students may just as easily discover other approaches (defining variables and writing equations; graphing, etc.) Problems 6-10 should give an opportunity to explore interesting aspects of comparing quantities--they require adding equal amounts, subtracting equal amounts, or adding and subtracting different amounts in search of solutions. A word of caution: problem 10 introduces the possibility that an answer might be negative (Tony and Jon suffer a loss of $15 each on their first day of business. Not all ideas are instant moneymakers!) It may be important to ask students how they can explain their answers and if they think their answers are the only possible ones for the problem conditions. Stretching studentss' grasp of the problem space will benefit from individuals or small groups sharing solutions with the class and reflecting together on the outcomes.

B. A Japanese Curriculum--Algebra Problems on the Internett

For those who have access to the Internet in their math labs or classrooms, there is a collection of Japanese algebra word problems online at: www.japanese-online.com/math. The problems presented are translated from a Junior High School math placement test of 225 problems that are given to 12 year olds. Students in Japan are given a time limit ranging from 1 to 5 minutes for each problem. To complete the 225 problems within the time provided requires over 8 hours of testing!

There are about 20 different types of word problems, so some selecting to arrange a set of cohesive problems addressing concepts would be needed by a teacher. However, this site could be used as randomly arranged to challenge students and provide some insight into the nature of math work students are doing in Japan.

Japanese students regularly place among the top 3 countries worldwide in student math abilities. (According to information on this website, the United States was recently ranked #14 in international math placement.)

Each problem on the site is presented with a clear (often colored) graphic and verbal description along with the minutes within which Japanese students are expected to solve the problem; most have multiple choice answers. In addition, each problem offers a "hint" button and an "explanation" button that provides the answer (most often with defined variables and equations). The site also offers a free download of World Math Challenge volume1. "The math problems contained within this site are for free use by individuals and can be copied for use by teachers within their classrooms."

This website provides such an accessible resource that all algebra teachers should check into using it for math tutorials or labs. It is limited to the defined variable and equation approach to solving problems, but very useful for students who need practice with reading and solving key types of problems. There is even a button to have the problem read aloud and a button to have the answer read aloud--this may make it possible for slower readers to do more challenging math. One problem from the website is included as Figure 5 in the appendix, to illustrate the arrangement visible on the screen when solving problems..

3. Lines, Diagonals, Handshakes and Sums as a Function of n items

I began this problem set with a question similar to one posed on a past CAPT math test: "how many diagonals can be drawn in a closed figure with *n* sides?" In working on this problem with the seminar group, it became clear that a whole series of problems, though seemingly unrelated, have somewhat similar mathematical structure--in other words, they share the same "problem space."

1. How many diagonals are there in a convex n-gon?
2. If there are n points in a plane (no 3 on the same line), How many lines can be drawn between all pairs of the points?
3. If there are n lines in a plane (no 3 through the same point), how many regions are formed?
4. If n people in a room all shake hands with each other, how many handshakes occur in total?
5. What is the sum of the first n numbers? (i.e., 1+2+3+4+5 . . .)
6. What is the sum of the first n odd numbers? (1=3+5+7+9 . . .)
7. How many ways can we express a positive number as the sum of two positive numbers if order is relevant? (i.e., for 3: 1+2 and 2+1 both count)
8. How many ways can we express a positive number as a sum of 2 positive numbers if order is irrelevant? (i.e., for 3: 1+2 is the same as 2+1)

A variety of approaches can be taken to solving these problems, but in keeping with the general theme of this unit--exploring the problem space--the most comparative approaches are best. By building a table of values for each problem simultaneously, patterns, similarities and differences emerge. (See Figure 6 in the appendix.))

These problems are ideal for “multiple solution” examination because they have similar progressions of numbers as functions of n. Students should be encouraged to sketch and determine values for at least n = 0 to 10 for each problem before attempting equation solutions. Encourage them to look at the patterns as they emerge. Some are so alike that students might gain new insights. By looking at the numbers for “lines” (problem2) and “handshakes” (problem 4), it readily is apparent that handshakes all around is the same as making sure each person (point) is connected to all others in a group. For a closed figure this amounts to all the sides plus all the diagonals. To determine the number of diagonals in closed figures (problem 1), students should look at the first order differences of numbers running down each column. The numbers for diagonals and lines both increase in a 2,3,4,5,6,7… sequence, but with different starting points of n. Graphing the tables of data would be another useful exercise for students to look at how these problems play out. Graphing calculators or using computer software with this set of problems is recommended.

It may be quite a leap to get to the equations for these problems, but that is the ultimate goal for students advancing in algebra. Sums of the first n odd integers (problem 6) should be not difficult at all. Discovering the similarities among the other seemingly different cases will be a challenging task requiring a good deal of time with the data. One strategy that we explored in our seminar involves representing the problems as stacks of blocks or grids. For example, problem 5, the sum of the first n numbers can be drawn as a rectangular grid of width n and length (n+1). Shading in half the blocks on this grid shows the table sequence (1,3,6,10, . . .) and demonstrates why the sum is just half of n times (n+1).

4. Unitary Rates and Mixing Problems

The problems in this set are intended to take students from simple, one step calculations to more complex problem solving involving unitary rates (e.g., price per pound) and mixing of rates. Each problem can be analyzed using a Vee Chart and using a horizontal unit conversion graphic organizer may assist solutions. (These solution techniques are demonstrated below.) Again, however, it is important to note that just finding the answer is less relevant to the lesson than the manipulating of the parts of these challenges to gain understanding of the nature of the concepts imbedded in mixed rate problems.

My students often find problems such as the Japanese Curriculum “hose and bucket” rate problem above logically possible to explain (“obviously the two hoses will fill the bucket faster than either one hose”), but mathematically they are at a loss at how to put the data together to get a concrete solution.

Sample solution for the “hose and bucket” problem:

The first step is to read carefully for details of the problem. Then these details need to be organized and examined for further less obvious information. For this I recommend a Vee - Chart technique, which is just a graphic organizer format for the problem solving analysis popularized classically by the mathematician George Polya.

(Chart available in print form)

Using the organizer for this problem hopefully will help students stretch their thinking from just the data presented in the problem to mathematical elaborations on that data--in this case forming unitary rates: the fraction of the total bucket that a hose can fill in one time unit (minute in this problem)..

Since the two hoses (A+B) can fill 1/45 + 1/30 of the bucket in one minute, the problem is one of finding a common denominator for 45 and 30. With minimal trial and error, most students should find that 90 works in this data. 1/45 = 2/90 and 1/30 = 3/90, so 1/45 + 1/30 = 2/90 + 3/90 = 5/90. 5/90 = 1/18, so the two hoses together can fill 1/18th of the bucket in 1 minute. Therefore, it takes 18 minutes to fill the whole bucket with both hoses. (A Japanese 12 year old is expected to solve this problem in just one minute!)

Similar strategies can be used with the problems below. Students need to see a variety of problems to gain understanding that unitary rates can be amount per 1 minute,

per 1 dollar, per 1 pound, etc., depending on the data in the problem

1. Cashews sell for $ 3.50 per pound (rate: $3.50/lb). How much will it cost to buy five pounds?
2. Almonds sell for $5.00 per pound (rate: $5.00/lb). How many pounds can be bought for $ 12.00?
3. What will it cost to buy two pounds of cashews (at $3.50/lb) plus one and one half pounds of almonds (at $5.00/lb)?
4. If three pounds of cashews are mixed with three pounds of almonds, what should be the total cost of the whole mixture?
5. What is the mixture price per pound for the mixture in problem 4?
6. If fifteen pounds of an almond/cashew mixture contain five pounds of almonds (at $5/lb), how many pounds of cashews (at $3.50/lb) does the mix contain?
7. What would be the total price for the fifteen-pound mixture in problem 6?
8. Equal amounts of cashews ($3.50/lb) and almonds ($5.00/lb) are mixed, what should be the price per pound for this mixture?
9. Mary wants to sell fifteen pounds of a cashew/almond mixture at the school fair for $4.00 per pound. How many pounds of each type of nut should be used?
10. James thinks they can sell more pounds of nut mix if they make a mixture that sells for $3.75 per pound. If they want to take in a total of $100 on the nut sale, how many pounds of mixture must be sold at $3.75/lb and what amount of each type of nut should be in the mix? Round answers to the nearest hundredth or penny.

5. Speed, Time, Distance Problems

In the Singapore 6A curriculum, algebra and speed problems form first and last chapters. Visual and graphic representations can be useful in solving these problems, but students may also be comfortable with tabular, graph or equation solutions (using the speed formula: speed = distance/time). In fact, since the speed, distance, time relationships are introduced at in elementary grades, this type of problems are ideal for vertical teams to consider and for students to apply to the "5-part problem-solving template."

The following Problem set starts with problems appropriate for middle school math and leads to problems that can challenge Algebra I students at the high school level. Data from the first problems is needed for later problems, so these should be solved in order.

1. In one hour, Anne's boat floats three miles down the Quinnipiac River. How fast is the river flowing?
2. If Anne can row 2 miles in one half hour on a still lake, what is her rowing speed in miles per hour?
3. Anne leaves her dock and tries rowing her boat upstream against the river current for one hour. How far upstream does she go in one hour?
4. When Anne turns and rows downstream with the current, how fast is she going?
5. How much time does it take Anne to row back to her dock from her farthest point upstream?
6. If Anne leaves her dock and rows downstream for one hour, how far will she then be from her dock?
7. How long will it take Anne to row back upstream to her dock this time?
8. Aaron puts a small motor on Anne's boat and goes six miles upstream in one hour. How fast would his motorboat go in still water?
9. If Aaron motors downstream for one hour, how far will he travel?
10. After going downstream for one hour, how long does it take Aaron to return upstream to his starting point?
11. Write the following ratios:

a. Anne's speed in still water to the speed of the river current
b. Aaron's boat speed in still water to the River current
c. Anne's upstream rowing speed to her downstream rowing speed
d. Aaron's downstream speed to his upstream speed

12. Aaron and Anne borrow a friend's canoe and paddle together on the West River. If it took an hour for them to go six miles upstream and only forty five minutes to return to their starting point, what is the speed of their boat in still water and what is the speed of the current in the West River?

6. Algebra Problems that use the Addition Table

"If this is algebra, why are we looking at the addition table?"

This will likely be the first response of your students to a suggestion of exploring problem space with the addition table. Students usually are convinced that addition and addition tables are only for the very young, even though they have rarely taken a look at the table with an eye to discover patterns and problem solving relationships. I include this topic here because we spent a few enjoyable hours in our seminar thinking mathematically about the addition (and multiplication) tables as launch pads for investigation into "big ideas" of math. For purposes of illustration, I include examples of problems which can be solved with tables and hope other math teachers will reconnect their students with these tools for thought and exploration.

Example 1. A student was tossing two 6-number cubes (dice) and wanted to know what chance she had of getting a number greater than nine.

By looking at a portion of the addition table (6 by 6) it is quite easy to see that there are 36 possible outcomes for the tossing of the dice and that 6 of those outcomes are greater than nine. Since the probability is a ratio of desired outcomes to possible outcomes for an event, her answer is 6/36 or 1/6.

A geometric grid aspect of this solution (see Figure 7 in the appendix) would lend itself to a variety of two dimensional problems involving various types of number cubes--all of which rely on the addition table format for "possible" outcomes versus "desired" outcomes.

Example 2. John was tossing a 4-sided number cube numbered 1-4 and a 7-sided number cube numbered 1-7 (such cubes are available in math supply catalogues). He wondered what his chances were of getting a number less than seven. By using the table in figure 8, determine his probability.

Again, the solution (1/2 or 50%) can quickly be determined from the portion of the addition table appropriate for this problem, or by using the corresponding grid (see Figure 8 in the appendix) to determine the ratio of desired outcomes (14) to total possible outcomes (4 x 7 = 28).

Any combination of two dimensions can be used (2x9, 3x3, 10x5, etc) to quickly work out the probable outcomes for a desired event.

7. A Political and Economic Problem?

As a final illustration of the remarkable possibilities for exploring problem space and mathematical concepts, I include a problem our seminar tackled which opens the door to interdisciplinary study of history, social studies, economics, ethics, and justice issues. The data in this hypothetical problem echo very real data recently debated in the popular press, which may give the mathematical understanding a bit more urgency and interest.

Problem I: A group forms 10 % of a larger (whole) group. The members of this group receive 60% of the income of the whole. How much richer, on average, are the members of the 10% group than the average of the rest?

A visual graphic representation of the problem (Appendix, figure 9) helps lead to the solution:

a. For the small population group:

Income % / Population % = 60% / 10% = 6

(each 1% of population in this group gets 6% of whole $)

b. For the majority:

Income % / Population % = 40% / 90% = 4/9

(each 1% of population in this group gets 4/9% of whole $)

c. To compare a & b:

6 / 4/9 = 54 / 4 = 13.5

So, the 10% group, on average, gets 13.5x the income of the 90% population group. Or, to be more dramatic numerically, each small group member gets, on average, 1350% of the average income of each large group member.

Problem II: Let's suppose that the "Whole Population " of our first problem represents just 5% of the World Population; and lets suppose the "Whole Income" of the first problem is 40% of the Worldwide Income.

The original 10% of the whole Population now is (10% of 5%) = .5% of the World Population and this group gets (605 of 40%) = 24% of the Whole World Income.

How much richer, on average, are members of the original "10% group" than the average for the rest of the world?

a. % of World Income/ % of World Population = 24%/ .5% = 48

b. For the rest of the world: % of World Income / % of World Population = 76% / 99.5% = .76

c. To compare a & b:

48/.76 = 63

Each member of the small group gets 63 times (6300%) of the average Worldwide person's income. Also, the members of the 90% group in problem I receive 3.6 times or 360% of the average worldwide person's income

The problem space just beginning to be explored in this case is the enormous issue of wealth and income distribution within a rich country and for that country in comparison to the rest of the world. As we enter a hot political season in this country, a great deal will be said about our economy and our world relations. Mathematical understandings will be needed to help cooler heads prevail in the midst of hot rhetoric. For those who have the opportunity to teach interdisciplinary courses, this problem may point the way to explore some critical issues in social studies with mathematical insights.

Bibliography for Teachers and Students

Berger, Debra Wolk, *Getting Ready to Take the Math 10 CAPT Test* . Brewster, New York: Kingsbridge, 2003.

The preparatory workbook many high schools in Connecticut are providing for students to cram for the tenth grade CAPT test in mathematics. A wide range of algebra and geometry problems is included and samples are provided with solutions for each section of the book. The nature of this and any test-prep book is that it emphasizes getting the answer more than understanding the nature of the mathematics. However, some interesting problems are included which could lead students to other investigations. In the short run, we are going to use these books, so mine them for valuable problems. The tenth grade CAPT test always has some great problems that stretch student thinking. All the "released" math questions from previous years of the test are available on the Connecticut State Department of Education's website.

Carpenter, Thomas, Megan Loef Franke, and Linda Levi. *Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School* . Portsmouth: Heinemann, 2003.

Discussion by teachers actively engaged in classrooms about what it means to do and learn math. Filled with examples that illustrate the algebraic thinking processes, which are (or should be) a part of the learning of arithmetic at the earliest grades. Of all books read in our seminar, this one opened up the most discussion among teachers from grade one to high school. Also includes a CD featuring classroom videos of students engaged in the problem soling dialogues described by the authors.

Charles, Randall, and Carne Barnett, *Problem-Solving Experiences in Pre-Algebra* . New York: Addison-Wesley, 1992.

A workbook of "thinking activities, problem solving strategies, challenge problems and more--designed to support and encourage student problem solving in math classes. Covers a wide range of math problems but requires careful selecting by teacher to develop focused problem sets around a key concept..

Harnadek, Anita, *Algebra Word Problems* . Pacific Grove, CA: Midwest, 1988.

A Set of twelve slim paperback booklets, each with a theme or concept (though three of the booklets re "miscellaneous" problems) around which the 40 to 50 problems are built. The explanations are spare and drive mostly toward variable definition and equation solving. There is no reason why students could not use my "5-part problem template" for any of the problems. This resource is an inexpensive way to obtain problem sets for some algebra topics and the problems provided generally progress from less to more difficult.

Kordemsky, B. The Moscow puzzles. Edited by Martin Gardner. New York: Dover, 1992.

Mathematical puzzles and problems often appearing simple, but proving quite tricky. Several of these nicely demonstrate common pitfalls of algebraic problem solving and could be well worth the time spent working through with students.

Lampert, Magdalene. *Teaching Problems and the Problem of Teaching* . New Haven: Yale University Press, 2001.

Detailed reflections by an experienced classroom teacher and university professor who spends a year teaching a fifth-grade math class as part of her research. All teachers of math can benefit from this thoughtful and well-written examination of the dynamics of teaching and teacher-student relationships. Particularly helpful is Lampert's careful attention to student problem solving by individuals and her whole class.

Loyd, Sam *. Mathematical Puzzles* . Selected and Edited by Martin Gardner. New York: Dover, 1959.

Puzzles originally published in magazines and newspapers by Sam Loyd, “America’s greatest puzzlist” have been organized in this volume by mathematical category. Many of these puzzles are classic word problems that can fascinate curious students or drive them crazy trying to solve them. Fortunately, solutions are provided in the appendix. With some guidance, many of the puzzles might be launching pads for exploration of mathematical big ideas.

Olas, Carla, *Problem Solving and Algebra Too* . Boston: Northeastern University Custom, 1992.

An algebra course workbook/practice book, this text has hundreds of problems with answers provided in the appendix, but little to no explanations. The author does provide problems sets with instructions to read them and sort into groups of related problems--a key step in developing the ability to explore “problem space.” This book may serve as a source of problems for a particular type of problem, but it is a bit overloaded with tedious black and white graphics and unhelpful details.

Polya, George, *How To Solve It* . Princeton: Princeton U. Press, 1973.

One of the classics in the writing about problem solving, this book offers a problem analysis that boils down to four steps: understand the problem, devise a plan, carry out the plan, and examine the solution. Vee-charts of word problems as I use them are graphic organizers based roughly on Polya’s steps.

Primary Mathematics Texts 6A and 6B , U.S. Edition. Curriculum Planning and Development Division, Singapore Ministry of Education. Singapore: Federal Publications, 2003.

As mentioned in my paper, this set of texts is a very accessible presentation of problems for pre-algebra to early algebra students. the arithmetic understandings developed will give a foundation for more advanced algebra. The “concrete--pictorial--abstractct’’ approach is helpful for students who struggle with too much perceived abstractness in mathematics.

Reimer, Wilbert, and Luetta Reimer, *Historical Connections In Mathematics: Resources for Using History of Mathematics in the Classroom* . Fresno: AIMS Educational Foundation, 1992.

A 3 volume series of books that use great mathematical thinkers to present historical connections as well as the key concepts the pioneer thinkers developed. Very useful for exploring the roots of some of the more interesting problem solving techniques (Napier’s Bones, for example).

Santi, Terri, *Math Ties: Problem Solving, Logic Teasers, and Math Puzzles* . Pacific Grove: Critical thinking books, 1998.

A great source of quick puzzles and problems to introduce new units or build into problem sets--simpler and more accessible than the Sam Loyd or Moscow Puzzles..

Smart, Margaret, and Mary Laycock, *Hands-On Math for Secondary Teachers* . Hayward, CA: Activity Resources, 1984.

Though somewhat older, this guide is an interesting resource for teachers who are seeking alternatives and different strategies for teaching fundamentals of math to junior and senior high school students. Of great potential are the use of graphic solutions to problems and use of simple drawings and manipulatives to make concepts real.

Websites:

www.japanese-online.com/math

As mentioned in my paper, this site provides 225 selections of 20 different types of algebra word problems along with illustrations, hints, explanations, and suggested time expectations for solving by 12-year-old students in Japan. Teachers and students can use

this site as a learning or tutoring tool.

www.mathpower.com

Author and math professor Ellen Freedman has created this site to offer basic math and algebra tutorials, word problems and sets of problems for students and teachers. The site uses humor and lively graphics to address concerns about math anxiety and to provide practice in solving problems. Links to nearly one hundred math and other educational websites are included.

Appendix

Figure 1: Sample Solution using a 5-part “multiple solutions” template

Pattern /Sketch

X

XX

XXX

XXXX

XXXXX

XXXXXX

Table of Values

# of Rows Blocks/row Total blocls

1 1 1

2 2 3

3 3 6

4 4 10

5 5 15

6 6 21

7 7 28

8 8 36

9 9 45

10 10 55

Verbal Description

Amir is arranging blocks. He puts one in the top row, 2 in the 2nd, 3 in the 3rd, etc. How many total blocks does he need to build 5 rows?

Equation

Let R = number of Rows

Let T = Total number of blocks

T = R + (R-1)R/2

or

T = 1/2 R(R+1)

Graph

(Graph is available in print form)

Figure 2. Multiple Solutions 5-part Problem Solving Template

Pattern /Sketch

Table of Values

Verbal Description

Equation

Graph

Figure 3. Singapore Approach Sample #1 Solution:

Ted |1 unit|

Jose | | |

After spending money:

Ted | | $20 |

Jose | | |

/-------- ($50) ----------- /

1 unit = $50 - $20 = $30

Ted started with $30

Jose started with 2 units = 2 x $30 = $ 60

Figure 4. Singapore Approach Sample #2 Solution.

After the grandmother's gift:

Sam | | |

Kim | |

Before the gift:

/ ---------- $140 ------------>/

Sam | | | (gift) |

Kim | | (gift) |

/---------$60------>/

|

\/

/ ---------------$140 ------------>/

| | | | |

/-----$60------>/ /---$60----->/

$140 - $60 - $60 = $20

The grandmother gave each of them $20.

Figure 5. A sample algebra problem form www.japanese-online.com/math

(All problems start with a sketch, but these do not download)

Hose “A” takes 45 minutes to fill the bucket with water. Hose “B” can do the same in 30 minutes.

If you use both hoses, how long will it take to fill the bucket?

You would have 1 minute to finish this problem if you were a Japanese 12 year old.

ANSWERS (Click the letter choices below to check your answer.)

12 minutes 18 minutes 16 minutes 13 minutes 21 minutes

HINT EXPLANATION NEXT QUESTION BACK TO INDEX

Press here to listen to the question.

Figure 6. Diagonals, Lines, Handshakes and Sums table

n	Diagonals	Lines	Regions	Hand-shakes	1st #s sum	1st Odd #s sum	Ordered sums of 2	Unordered sums of 2
0	-	-	1	-	-	-	-	-
1	-	0	2	0	1	1	0	-
2	-	1	4	1	3	4	1	1
3	0	3	7	3	6	9	2	1
4	2	6	11	6	10	16	3	2
5	5	10	16	10	15	25	4	2
6	9	15	22	15	21	36	5	3
7	14	21	29	21	28	49	6	3
8	20	28	37	28	36	64	7	4
9	27	36	46	36	45	81	8	4
10	35	45	56	45	55	100	9	5

.

.

.

n n(n-3) /2 n(n-1) /2 n(n+1) +1/2 n(n-1) /2 n(n+1) /2 (n x n) (n-2)(n-1)/2

Figure 7. Addition table and grid for probability problem solving, Example 1.

1 2 3 4 5 6

1 2 3 4 5 6 7

2 3 4 5 6 7 8

3 4 5 6 7 8 9

4 5 6 7 8 9 **10**

5 6 7 8 9 **10 11**

6 7 8 9 **10 11 12**

Figure 8. Addition table and grid for probability problem solving, Example 2.

1 2 3 4 5 6 7

1 **2 3 4 5 6** 7 8

2 **3 4 5 6** 7 8 9

3 **4 5 6** 7 8 9 10

4 **5 6** 7 8 9 10 11

Figure 9. Graphic illustration for a "political and economic" problem.

(figure is available in print form)

Shaded = 10% of population

Shaded = 60% of Total Income

Endnotes

1. "Fewer noses Stuck In books In America, Survey Finds," Times [New York] 8 July 2004. Reporting on the analysis by The National Endowment for the Arts of data from The Survey of Public Participation in the Arts conducted by the Census Bureau in 2002.

2. Vertical TeamTM is a term trademarked by The College Board Pre-AP program

3. Michelle Ralston handout on "Vertical Teams Strategies and Implications." New Haven Public Schools Workshop, 4 November 2003.

4. Ibid.

5. Ibid.

6. Professor Roger Howe, "Comments during a meeting of The Craft of Word Problems seminar", Spring, 2004.

7. Thomas Carpenter, Megan Loef Franke and Linda Levi, Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School (Portsmouth: Heinemann,

8. 2003) xi.

9. Thomas Carpenter, Megan Loef Franke and Linda Levi, Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School (Portsmouth: Heinemann, 2003) 1.

10. Thomas Carpenter, Megan Loef Franke and Linda Levi, Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School (Portsmouth: Heinemann, 2003) xii. Adapted from a vertical team "4-part problem page" presented by Michelle Ralston.

11. Primary Mathematics Texts 6A , U.S. Edition. Curriculum Planning and Development Division, Singapore Ministry of Education. (Singapore: Federal Publications, 2003) 3.

12. Thomas Carpenter, Megan Loef Franke and Linda Levi, Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School (Portsmouth: Heinemann, 2003) 134.

13. Ibid.

14. Thomas Carpenter, Megan Loef Franke and Linda Levi, Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School (Portsmouth: Heinemann, 2003) 137.

15. www.japanese-online.com/math