Yale-New Haven
Teachers Institute ${ }^{8}$

# LESS is MORE: Realizing Mathematics through Architecture 

Curriculum Unit 06.04.05
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## Introduction

The purpose of this proposal is to introduce and integrate architecture as a study subject in the math curriculum for high school grade levels within the public school system at New Haven, CT. Students will conclude this program by creating an architectural model that demonstrates the mathematical discoveries of their work and study. While discoveries are in the traditional areas of Algebra, Geometry, Trigonometry, and Calculus, the process of learning and teaching adheres to the architectural paradigm of "...less is more."[1]

To empower students with those architectural principles that delineate an historic relevance for the precept that "...less is more," this curriculum unit combines a series of presentations that explore the formal definitions of geodesic geometry in Architecture and, in particular, the engineered geometry, patterns, and systems that modularize those design forms. The dynamics of teaching and learning with these tangible forms is intended to enhance the visual and intellectual perceptions of young mathematicians, as they contend with less intangible concepts. Therefore, the educational goals strive to reach a more profound level of awareness and with less compromise. Teaching the project presented in this unit describes my experiences in organizing, managing, supervising, and testing this program. To clarify the development process of this project, three (3) phases will be presented as follows:

## Phase 1: Background and Program Description

This phase describes the project-launch, establishes criteria for the selected students, and a bibliography for student readings. This phase also describes the selected architecture that will be compared and contrasted, promoting mathematical axioms and theorems integrated within the mathematics curricula. Buckminster Fuller and his many inventions will be introduced in this phase.

## Phase 2: Implementation Models

This phase will be a general overview and introduction to architecture and architectural engineering. Students will be introduced to the architectural and engineering principles that the selected structure exemplify. Site locations will span the global and include several in New York City for onsite visits. Some of these principles will further associate with a variety of interdisciplinary teachings, in English, History, or Science.

## Phase 3: Deployment and/or Execution of the Program

The third phase will explain how the project will be executed. The program will be introduced to the students in two parts. In the first part students will review some of the more popular sites. This will be by Internet content. The second part will be dedicated to a hands-on approach. Students will collect all their data learned during the program and then, they will create an architectural model that demonstrates the mathematical discoveries of their work and study. Study forms and shapes include: tetrahedrons; octahedrons; icosahedrons; and cuboctahedrons.
(Recommended for Mathematics (Algebra, Calculus, Geometry, and Trigonometry), grades 9-12.)
Image reference: http://www.kwsi.com/ynhti/images/image01

## Phase 1: Background and Program Description

This program will enable students to respond to a series of sequential assignments that culminate and terminate in one or more definitions of a geodesic dome and its components. These definitions will be in Algebra, Calculus, Geometry, Trigonometry, or some combination thereof.

A geodesic dome is an almost spherical structure based on a network of struts arranged on great circles (geodesics) lying on the surface of a sphere. The geodesics intersect to form triangular elements that create local triangular rigidity and distribute the stress. It is the only man made structure that gets proportionally stronger as it increases in size.

Of all known structures made from linear elements, a geodesic dome has the highest ratio of enclosed volume to weight. Geodesic domes are far stronger as units than the individual struts would suggest. It is common for a new dome to reach a "critical mass" during construction, shift slightly, and lift any attached scaffolding from the ground.

Geodesic domes are designed by taking a Platonic solid, such as an icosahedron, and then filling each face with a regular pattern of triangles bulged out so that their vertices lie in the surface of a sphere. The trick is that the sub-pattern of triangles should create "geodesics", great circles to distribute stress across the structure.

There is reason to believe that geodesic construction can be effectively extended to any shape, although it works best in shapes that lack corners to concentrate stress.

## The Math Lessons

## a) The Algebra of a Geodesic Dome.

Although the Geometry lesson below explores both the necessary and sufficient conditions for tensegrity equilibria, static models of tensegrity structures reduce to linear algebra problems. After first characterizing the problem in a vector space where direction cosines are not needed, the components of all member vectors are described. While our approach enlarges (by a factor of 3) the vector space required describing the problem, the advantage of enlarging the vector space makes the mathematical structure of the problem amenable to linear Algebra treatment. Using the linear algebraic techniques, many variables are eliminated from the final existence equations however; the Geometry approach most fully addresses the understandings of Tensegrity.

## b) The Geometry of a Geodesic Dome.

The Geometry lesson characterizes the necessary and sufficient conditions for tensegrity equilibria. Static models of tensegrity structures are reduced to linear algebra problems, after first characterizing the problem in a vector space where direction cosines are not needed. This is possible by describing the components of all member vectors. While our approach enlarges (by a factor of 3 ) the vector space required describing the problem, the advantage of enlarging the vector space makes the mathematical structure of the problem amenable to linear Algebra treatment. Using the linear algebraic techniques, many variables are eliminated from the final existence equations however; the Geometry approach most fully addresses the understandings of Tensegrity.

Tensegrity is the pattern that results when push and pull have a win-win relationship with each other. The pull is continuous and the push is discontinuous. The continuous pull is balanced by the discontinuous push producing an integrity of tension-compression.

Push and pull seem so common and ordinary in our experience of life that we humans think little of these forces. Most of us assume they are simple opposites. In and out or back and forth, force that is directed in one direction or the opposite direction.

Fuller explained that these fundamental phenomena were not opposites, but complements that could always be found together. He further explained that push is divergent while pull is convergent.

Imagine pushing a yellow ping pong ball on a smooth table with the point of a sharp pencil. The ball would always roll away from the direction of the push, first rolling one way then the other. Push is divergent. Now imagine the difference, if you attach a string to the ping pong ball with tape, and pull it toward you. No matter
how other forces might influence the ball to roll away from you, the string would always bring it to you more and more directly. Pull is convergent.

Image reference: http://www.kwsi.com/ynhti/images/image02

## PUSH

Image reference: http://www.kwsi.com/ynhti/images/image03

## DIVERGES

Image reference: http://www.kwsi.com/ynhti/images/image04

## CONVERGES

Image reference: http://www.kwsi.com/ynhti/images/image05

## PULL

Another example from another common experience occurs when pulling a trailer with a car. While pulling uphill, the pull is against gravity; and the trailer converges smoothly behind my car. If the trailer begins to sway, an increase in pulling by accelerating will dampen the sway. In comparison, if the trailer begins to sway while traveling downhill, the trailer may begin to push. This produces a strong side to side force, or divergence. The trailer will also begin to sway from side to side. This push is divergent. When the trailer begins to push us, acceleration will re-establish pull. This pull is convergent. The trailer will respond by straightening and regain the original equilibrium. Therefore, contrasting fundamentals always co-exist in pairs: Push and Pull; or Compression and Tension; or Repulsion and Attraction.

## c) The Trigonometry of a Geodesic Dome.

Drawing a Grid Diagram
Image reference: http://www.kwsi.com/ynhti/images/image06
(1) Draw an equilateral triangle.
(2) Subdivide the edges into N parts for N frequency.
(3) Join the points of subdivision with a 3 way grid.
(4) Start at the top vertex and number all the crossing points for the frequency $N$ (the 1st point will be 0,0 and the last point will be $N, N$ ) every point has a two (2) number designation.
(5) Draw the 3 medians of the face triangle.
(6) The medians describe 6 right triangles, each containing the whole symmetry system of the polygon. This pattern is the basic quanta of this system. The structure consists of repetition of this.
(7) If the structure considered is spherical, chord factors are only needed for the break down edges that lie partly or completely within the symmetry triangle.
(8) For a sphere, the only coordinates required are the end points of the edges contained by the symmetry triangles.
(9) The system repeats symmetrically.
(10) If the top left symmetry triangle is used the numbers to be manipulated are handier as many values are 0.

## Explaining the Geodesic Algorithms

The geodesic algorithm is the mathematical procedure for finding the strut lengths of a geodesic dome. This algorithm utilizes spherical trigonometry to solve for spherical coordinates of the vertex points of a facet diagram for a given frequency and radius dome. Then the chordal distances between adjacent vertices are computed using the spherical coordinate distance formula. These distances are the strut lengths.

Solving for the vertex coordinates involves solving a series of 4 spherical triangles.
Image reference: http://www.kwsi.com/ynhti/images/image07
Diagram 1: Begin with a spherical icosahedral facet (spherical triangular side) with dimensions of 72 degrees at each vertex angle ( $\mathrm{A}, \mathrm{B}$ and C ) and sides ( $\mathrm{a}, \mathrm{b}$ and c) of 63 degrees, 26 minutes and 06 sec . (Note: in spherical trig, sides are dimensioned in terms of angles with respect to dome center.) With a frequency of $F$, the sides are divided into F parts; so that Sides b' and a' are known (selected so that the vertex to be calculated is intersected) as well as Angle C. Using spherical trig equations 1, 2 and 3 , Angle 0 is found for use in Solution 2. These 3 equations and equation 4 embody the Side-Angle-Side (SAS) Subroutine.

Image reference: http://www.kwsi.com/ynhti/images/image08
Diagram 2: Angle $A$ is known to be 72 Degrees and $b^{\prime \prime}$ and $c^{\prime \prime}$ are known (selected so that the vertex to be calculated is intersected ). Angle y is to be found, so again equations 1,2 and 3 (the SAS Subroutine) are used to solve for y . Both angles o and y are used in Diagram 3 (Solution 3).

Image reference: http://www.kwsi.com/ynhti/images/image09
Diagram 3: Using Angles o and $y$ and the difference between Sides $b^{\prime}$ and $b^{\prime \prime}$ (= Side $b^{\prime \prime \prime}$ ) we solve for $X$. This time, equations 5, 6 and 7 (the Angle-Side-Angle (ASA) Subroutine) are used to obtain Side X.

Image reference: http://www.kwsi.com/ynhti/images/image10
Diagram 4: The final step in obtaining the spherical coordinates for a given vertex is from Side $b^{\prime}$, Side $X$ and Angle o from which we can obtain Angle Z (using Eq. 1, 2 and 3) and Side Y (using Eq. 4 ) as the coordinates of point $P$.

The coordinates for adjacent vertices are applied to Equation 8 to obtain the strut lengths, which are the chordal distances between the vertices.

## Spherical Trigonometry Equations

Note: Both angles and sides should be measured angularly in spherical trig.

## SAS

With equations 1,2 and 3 an angle ( 0 ) can be found given 2 sides ( $A$ and $B$ ) and the included angle $B$
(1) $o+d=2 \arctan ([\cos 1 / 2(a-b) / \cos 1 / 2(a+b)] \cot B)$
(2) $0-d=2 \arctan ([\sin 1 / 2(a-b) / \sin 1 / 2(a+b)] \cot B)$
(3) $0=[(0+d)+(0-d)] / 2$
the third side (C) is solved for using eq. 4 after solving eq. 1, 2 and 3
(4) $c=2 \arctan ([\sin 1 / 2(o+d) / \sin 1 / 2(o-d)] \tan 1 / 2(a-b))$
(where $o, d B$ are angles ; $a, b$ and $c$ are sides)

## ASA

With Equations 5, 6 and 7 a side (a) can be found given 2 angles ( 0 and d) and the included side (c)
(5) $a+b=2 \arctan ([\cos 1 / 2(0-d) / \cos 1 / 2(0+d)] \tan 1 / 2 c)$
(6) $a-b=2 \arctan ([\sin 1 / 2(0-d) / \sin 1 / 2(0+d)] \tan 1 / 2 c)$
(7) $a=[(a+b)+(a-b)] / 2$
(where $\mathrm{a}, \mathrm{b}$ and c are sides ; o and d are angles)
From the first 7 equations, the polar coordinates of each point on a grid can be derived. These coordinates can then be plugged into the Chordal Distance Formula (Equation \#8) yielding the strut (chord) length between 2 points on a sphere.
(8) $D=\left[r(1)^{* * 2}+r(2) * * 2-2 * r(1)^{* r}(2) *(\cos (d(1)) * \cos (d(2))+\cos (o(1)-o(2)) * \sin d(1) * \sin d(2)]^{\wedge} 1 / 2\right.$
in the spherical case $r(1)=r(2)=1$.

## Chord Factors and Base Ratio

After the strut lengths have been found, the repetitive pattern within the facet is established by graphical construction of the medians of the facet and numerical balancing of strut length distribution to obtain overall symmetry within the facet while maintaining the major arc lengths. Balancing is actually averaging; for each strut position in the symmetry triangle take the average of all strut lengths in the facet at that particular locus.

The resulting balanced strut lengths are then reduced to the most general form called the base ratio (Chord Factor) by dividing the strut lengths by the icosa-edge strut length. This yields a value of 1 for icosa-struts and a slightly higher value for the others. This base ratio is easily remembered and is used for any radius dome of the particular frequency. After the base ratios have been derived, further use of the geodesemetry algorithm is unnecessary and even ill advised because it is not symmetrical (also much slower ).

There is a set of unique Base Ratios (also called Chord Factors) for each frequency, regardless of the radius.

Uniform dimensions, chord factors and ratios may be listed for any size dome. The only numerical variable in geodesic spherical structures is that of the systems radius.

The name chord factor is assigned to all the constant lengths of a spheres connecting lines whether between any 2 spherical surface lengths or between 2 concentric spheres that are inter-triangularly trussed (or related structuring, i.e. door openings).

The spherical surface angles of the sphere and the central angles may be expressed in the same decimal fractions - which remain constant for any size sphere since they are fractions of a unit finite whole system.

Central angles of great circles are defined by 2 radii, the outer ends of which are connected by both an arc and a chord- which arc and chord are directly proportional to each unique such central angle.

The chord and the 2 radii form an isosceles triangle. The distance between the mid-arc and the mid-chord is called the arc altitude.

The frequency of modular subdivision of the edge of the icosahedrons facets may be multiplied at will once the spherical trigonometry rates of change of central and surface angle subdivisions have been solved. This is the essence of geodesic structures. In order to facilitate interchangeability of struts between domes, domes of consecutive frequencies can be based on a standard icosa-edge size. Within a series, icosa-edge struts remain the same and the radius divided by the frequency is constant (i.e. series number = radius/frequency or series $x$ freq. $=$ rad. ). For example, consider the \#4 Series with a 4.43 Icosaedge strut: $4 \times 2$ freq. $=8$ rad. , $4 \times 3$ freq. $=12^{\prime}$ rad. , $4 \times 4$ freq. $=16$ ' rad. , etc. Building domes of a given series makes for interchangeable parts while disregarding or switching series means more and different strut sizes.

## d) The Calculus of a Geodesic Dome

A geodesic is a locally length-minimizing curve. Equivalently, it is a path that a particle which is not accelerating would follow. Within each plane, the geodesic permutations are straight lines. Across the sphere, the geodesic permutations are large circles (like the equator). Whereas the geodesics within space depend on the Riemannian metric; and also affects the interpretations of distance and acceleration.

Geodesics preserve a direction on a surface (Tietze 1965, pp. 26-27) and have many other interesting properties. The to any point of a geodesic arc lies along the normal to a surface at that point (Weinstock 1974, p. 65).

Furthermore, no matter how a is distorted, an infinite number of closed geodesics exist on it. This general result, demonstrated in the early 1990s, extended earlier work by Birkhoff, who proved in 1917 that there exists at least one closed geodesic on a distorted sphere, and Lyusternik and Schnirelmann, who proved in 1923 that there exist at least three closed geodesics on such a sphere (Cipra 1993, p. 28).

For a surface defined parametrically by $\boldsymbol{x}=\boldsymbol{x}(\mathbf{u}, \mathbf{v}), \boldsymbol{y}=\boldsymbol{y}(\mathbf{u}, \mathbf{v})$, and $\boldsymbol{z}=\boldsymbol{z}(\mathbf{u}, \mathbf{v})$, the geodesic can be found by minimizing the arc length
(formulas available in print form)
A surface of revolution is a surface generated by rotating a two-dimensional curve about an axis. The resulting surface therefore always has azimuthal symmetry. Examples of surfaces of revolution include the apple, cone (excluding the base), conical frustum (excluding the ends), cylinder (excluding the ends), Darwin-de Sitter
spheroid, Gabriel's horn, hyperboloid, lemon, oblate spheroid, paraboloid, prolate spheroid, pseudosphere, sphere, spheroid, and torus (and its generalization, the toroid).

For a surface of revolution in which $\boldsymbol{y}=\boldsymbol{g}(\boldsymbol{x})$ and is rotated about the x -axis so that t
(formulas available in print form)

Phase 2: Implementation Models
Build a Geodesic Dome Model
One certain way to understand the characteristics and construction of a geodesic dome is to build a model of one. The following directions produce a low-cost, easy to assemble model of one type of geodesic dome. The triangular panels described below should be constructed from heavy paper or transparencies, and then connected with paper fasteners or glue.

Geodesic domes are usually hemispheres (parts of spheres, like half a ball) made up of triangles. The parts of a triangle are called the face (the part in the middle), the edge (the line between corners), and the vertex (where the edges meet). All triangles have two faces (one viewed from inside the dome and one viewed from outside the dome), three edges and three vertices. There can be many different lengths in edges and vertexangles in a triangle. All flat triangles have vertices that add up to 180 degrees however, triangles drawn on spherical surfaces or other shaped surfaces do not have vertices that add up to 180 degrees. Geodesic domes require that all the triangles are flat.

Image reference: http://www.kwsi.com/ynhti/images/image11
Image reference: http://www.kwsi.com/ynhti/images/image12
One kind of triangle is an equilateral triangle, which has three edges of identical length and three vertices of an identical angle ( 60 degrees). There are no equilateral triangles in a geodesic dome, although the differences in the edges and vertices are not always immediately visible. This particular geodesic dome uses three different edge lengths and two types of triangles.

Edge Lengths: $\mathrm{A}=.3486 \mathrm{~B}=.4035 \mathrm{C}=.4124$
The edge lengths listed above can be measured in any way you like (including inches or centimeters); what is important is to preserve their relationship. For example, if you make edge A 34.86 centimeters long, make edge B 40.35 centimeters long and edge C 41.24 centimeters long. This dome has a radius of one: that is, to make a dome where the distance from the center to the outside is equal to one (one meter, one mile, etc.) you will use panels that are divisions of one by these amounts. So if you know you want a dome with a diameter of one, you know you need an A strut that is one divided by .3486.

The triangles can be constructed by their angles. Do you need to measure an AA angle that is exactly 60.708416 degrees? Not for this model: measuring to two decimal places should be enough. The full angle is provided here to show that the three vertices of the AAB panels and the three vertices of the CCB panels each add up to 180 degrees.

## Vertices:

$A A=60.708416 A B=58.583164 C C=60.708416 C B=58.583164$

1. The dome requires seventy-five (75) triangles with two $C$ edges and one $B$ edge. These are labeled CCB panels below because they have two $C$ edges and one $B$ edge. The dome also requires thirty (30) triangles with two $A$ edges and one $B$ edge, including a foldable flap on each edge for joining triangle-to-triangle with the chosen paper fasteners or glue. These panels are labeled AAB panels below, because they have two A edges and one B edge. The panel count per type should be: 75 CCB panels and 30 AAB panels.

Image reference: http://www.kwsi.com/ynhti/images/image13
Image reference: http://www.kwsi.com/ynhti/images/image14
2. Connect the $C$ edges of six CCB panels to form a hexagon (six-sided shape); the outer edge of the hexagon should be all B edges. Construct ten hexagons from six CCB panels. After this construction these hexagons are not flat; instead they are a very shallow dome.

Image reference: http://www.kwsi.com/ynhti/images/image15
3. From the remaining CCB panels construct five half hexagons in which the three $B$ vertices touch and four of the six $C$ edges touch.

Image reference: http://www.kwsi.com/ynhti/images/image16
4. Connect the $A$ edges of five $A A B$ panels to form a pentagon (five-sided shape); the outer edges of the pentagon should all be B edges. Create six pentagons of five AAB panels. The pentagons will also form a very shallow dome.

Image reference: http://www.kwsi.com/ynhti/images/image17
5. This geodesic dome is built from the top downward and outward. One of the pentagons made of AAB panels is will be the top. Connect five hexagons to one of the pentagons; the $B$ edges of the hexagons are the same length as the B edges of the pentagon, so the connection should be seamless. It should become apparent that the shallow curvatures of the hexagonal and pentagonal panels form a less shallow dome when joined together.

Image reference: http:/www.kwsi.com/ynhti/images/image18
Image reference: http:/www.kwsi.com/ynhti/images/image19
6. Connect five pentagons and to the outer edges of five of the hexagons. These connections should join the $B$ edges.

Image reference: http:/www.kwsi.com/ynhti/images/image20
7. Connect six hexagons to the outer B edges of the pentagons and the hexagons.

Image reference: http:/www.kwsi.com/ynhti/images/image21
8. Connect the five half hexagons to the outer edges of the hexagons.

Image reference: http:/www.kwsi.com/ynhti/images/image22
This geodesic dome comprises the equivalent of $5 / 8$ ths of a sphere, and is identified as a three-frequency (3) dome. The frequency of a dome is measured by the number of edges from the center of one pentagon to the center of the next pentagon. If the frequency of a geodesic dome is increased, by increasing the number of edges between neighboring pentagons, the spherical appearance of a dome is increased.

Students can explore the realities of dome-applications. Perhaps this dome could become a greenhouse, or house, or school, or corporate park. Perhaps it should be located beneath the ocean, or in outer space, even orbiting the Earth. In further detail, if this dome were a building: where would the doors and windows be located?

Alternatively, this dome can be constructed of struts, in lieu of panels, by maintaining the same length ratios referenced above. The required struts are: 30 type A struts; 55 type B struts; and 80 type C struts.

## Phase 3: Deployment and/or Execution of the Program

As in any sound lesson plan, assessments should be performed throughout the Phases of these lessons. Perhaps starting with demonstrating a student's comprehension of the parts and concluding with demonstrating a student's comprehension of the total dome is a rational approach to these metrics.

Therefore, students can be assigned the mathematical content and eventual construction of a panel, or a strut, or an entire geodesic dome. Each student will be assessed pursuant to a rubric. Rubric guidelines will contain scoring their calculations, their ability to perform cooperatively and individually. All should conform to NHSD, state and national standards.

## Appendix

a) A Student-generated Glossary from the following vocabulary words is recommended. (table available in print form)
b) Computer Programmed Geodesic Calculations

There are programs for many of the calculations required in designing geodesic domes (including the geodesic algorithm). Exploring these algorithms and the packaged programs provides an essential technology tangent to the more traditional teaching methods.

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http://www.kwsi.com/ynhti/images/image01
http://www.kwsi.com/ynhti/images/image02
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## DIVERGES

http://www.kwsi.com/ynhti/images/image03 CONVERGES
http://www.kwsi.com/ynhti/images/image04 PULL
http://www.kwsi.com/ynhti/images/image05
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