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Discovering Conic Sections in the Motion of Heavenly Bodies

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by Sam H. Jones

Introduction

The development of mathematics, from its very beginnings, has been about problem solving. Whether it has been commerce, trade or navigation we have always used early mathematical explorations of our world for very practical purposes.

The study of the heavens, in addition to the quest for knowledge about the world around us, was required in order for human beings to be able to better navigate, and map, the vast world around them. It is in this context, a practical applications approach, in which I would like to teach this unit.

Metropolitan Business Academy is an inter-district magnet high school in New Haven. The students have primarily chosen to pursue careers in business. Historically these students have done better on the verbal portion of standardized tests such as the CAPT. It is the intent of this unit to draw upon these superior verbal skills to teach about conic sections.

Rather than focus on the abstract and procedural nature of the subject we will attempt to put a human face on what is a very human endeavor. Additionally, we will emphasize the problem solving and practical aspects of the whole enterprise. By having students "discover" ways to solve these problems in a historical context we hope to facilitate a deeper and better understanding.

The seminar will allow me access to source materials, which will be invaluable in the preparation of this curriculum unit. The materials will be used to develop a curriculum, which is relevant to students of business in the practical application of solving real world problems.

The New Haven Precalculus curriculum covers Analytical Geometry, in general, and Conic Sections in particular, in unit VII. The unit specifically directs students to solve real world problems. As with many - if not most - mathematical applications, the math of conic sections was developed in conjunction with solving real world problems. Specific mathematics strands include:

1 Visualize three-dimensional objects from different perspectives and analyze cross-sections,

surface area, and volume. (3.2a1)

2 Explore conic sections and their applications graphically and symbolically (1.1a3) [1]

There will also be considerable attention on the business of mathematics as it relates to the scientific method. In particular, we will observe how data is collected, analyzed, and conjectures are then made.

Discovering conics

It is generally believed that Conic sections were first studied, in the abstract, by Euclid (around 300 BC) and later extended by Apollonius of Perga (around 200 BC) for no apparent practical purpose. Apollonius gave us the names of conic sections, which we still use today, ellipse, parabola, and hyperbola. Each is a cross-section of a cone (much like a paper cone for water at the doctor's office) which is sliced. Each curve, or cross-section, results from the intersection of a plane with a cone as if the plane is slicing the cone from varying angles as can be seen in Figure 1. [2]

(image available in print form)

Figure 1. *source:* Wikipedia

However the fact that Apollonius used his theories of conic sections to create more accurate sundials suggests that the following scenario may be more likely. During its daily course above the horizon the Sun appears to describe a circular arc. Supplying in his mind's eye the missing portion of the daily circle, the Greek astronomer could imagine that his real eye was at the apex of a cone, the surface of which was defined by the Sun's rays at different times of the day and the base of which was defined by the Sun's apparent diurnal course. Our astronomer, using the pointer of a sundial, known as a gnomon, as his eye, would generate a second, shadow cone spreading downward. The intersection of this second cone with a horizontal surface, such as the face of a sundial, would give the trace of the Sun's image (or shadow) during the day as a plane section of a cone. (The possible intersections of a plane with a cone, known as the conic sections, are the circle, ellipse, point, straight line, parabola, and hyperbola.)

However, compilers of the ideas in the history of the philosophy of science (known as doxographers) ascribe the discovery of conic sections to Menaechmus (mid-4th century BC), a student of Eudoxus, who used them to solve the problem of duplicating the cube. His restricted approach to conics--he worked with only right circular cones and made his sections at right angles to one of the straight lines composing their surfaces--was standard down to Archimedes' era.

A right circular cone is defined as a cone with a circle as its base and the apex is centered directly over the center of the circle so that the height from the base to the apex is at a right angle from the center of the circle to the apex. Figure 2 shows both a right circular cone (on the left) and an oblique circular cone (on the right). A cone may also have a non-circular base.

(image available in print form)

Figure 2. *source:* Wikipedia

Euclid adopted Menaechmus's approach in his lost book on conics, and Archimedes followed suit. Doubtless, however, both knew that all the conics can be obtained from the same right cone by allowing the section at any angle. Whereas his predecessors had used finite right circular cones, Apollonius considered arbitrary (oblique) double cones that extend indefinitely in both directions.

The reason that Euclid's treatise on conics perished is that Apollonius of Perga (c. 262-c. 190 BC) did to it what Euclid had done to the geometry of Plato's time. Apollonius reproduced known results much more generally and discovered many new properties of the figures. He first proved that all conics are sections of any circular cone, right or oblique. Apollonius introduced the terms *ellipse* , *hyperbola* , and *parabola* for curves produced by intersecting a circular cone with a plane at an angle less than, greater than, and equal to, respectively, the opening angle of the cone.

For this section there will be a lesson with a hands on activity where students will be constructing and also slicing cones. A sundial will also be demonstrated.

The Circle

The general formula for a circle centered at point (*h, k*) is given as:

$(x-h)^2 + (y-k)^2 = r^2$.

Although Ptolemy (around 100 AD) probably studied, and was familiar with, conic sections such as the ellipse, they were not part of his astronomy. It would be nearly two millennia from the inception of conic sections before conic sections, other than the circle, would play a role in astronomy.

Early observations led to the belief that the Earth did not move, and consequently must be the center of the universe. It is a natural conclusion since it is not readily apparent that the Earth is actually moving. After all, on a still day there is no wind or any other indication of motion.

The problem remained, however, of how to explain the motion of the heavens. As idealists, the Greeks were certain that the natural world must represent perfection. The perfect motion, or shape, about something is a circle. Unfortunately, observations of the heavens did not support the idea of the planets and stars rotating in perfect circles around the Earth. Problems, such as retrograde motion, varying speeds and apparent distance, could not be explained with this model.

In order to maintain the inherent perfection of this geocentric world view, with perfect circular motion, Ptolemy developed a system where the planets would travel in small circles (called epicycles) within a larger circle (called a deferent) around the earth. As complicated as it was, it was a better description than had previously been developed.

Copernicus (around 1500 AD) had the revolutionary idea that rather than the Earth being the center around which all heavenly things revolved it was actually the Sun, which was at the center. Instead of a geocentric universe, it was a heliocentric universe. According to the Copernican view, planets still moved in circular orbits, which meant that epicycles were still necessary to explain the observed motions of the heavens.

Although analytic geometry, as we know it, was not yet formally born we will introduce the definition and general equation of the circle.

A circle is the set of all points in a plane such that the distance (radius) from a given point (center of the circle) is constant. The general equation of the circle with center (*h, k*) and radius r is: $(x-h)^2 + (y-k)^2 = r^2$. Students should be able to write the equation of circles given identifying characteristics and graph the equations of circles.

The Ellipse

The general equation for the ellipse is given as: $x^2/a^2 + y^2/b^2 = 1$, *where* $b^2 = a^2 - c^2$

Tycho Brahe (around late 1500 AD) built what could be characterized as an early astronomical observatory. He is credited with the most accurate astronomical observations of his time. He created a huge database, which catalogued the stars and planets positions with great accuracy. Redundant observations allowed an accurate track of the heavenly bodies' motions even though the math had not yet been developed or applied in order to make accurate predictions.

Using Brahe's observations Johannes Kepler (around early 1600's AD) would develop his now famous laws of planetary motion. The first law states that the orbits of the planets are elliptical, with the sun at one of the foci. The second law states that a line joining a planet and the sun sweeps out equal areas during equal intervals of time. The third law states, the squares of the orbital periods of planets are directly proportional to the cubes of the semi-major axis of the orb.

An ellipse is the set of all points in a plane such that the sum of the distances (focal radii) from two given points (foci) is constant. The parts of the ellipse are shown in Figure 3. F1 and F2 are the foci. Line segment AB is the major axis. Line segment CD is the minor axis. The semi-major axis is denoted by the letter "a "in Figure 3 while the semi-minor axis is denoted by the letter "b". The eccentricity of an ellipse as in Figure 3 may be determined with the equation,

$E = (1- b^2/a^2)^{1/2}$

where *a* and *b* are the semi-major and semi-minor axes respectively.

(image available in print form)

Figure 3. *source:* Wikipedia

Students should be able to write and graph equations of ellipses, given their identifying characteristics.

The Parabola

The general equation of the parabola is given as: $y-k=a(x-h)^2$

In some sense the algebra for the parabola will perhaps be the most familiar to the students. However, they will not probably be familiar with the following definition. A parabola is the set of all points in a plane that are the same distance from a given point (focus) as they are from a given line (directrix). $y-k=a(x-h)^2$. This is pictured in Figure 4 where any point P will be equidistant from the focus F and the closest point Q on the directrix.

(image available in print form)

Figure 4. *source:* Wikipedia

The parabola may be illustrated with the trajectory of a rocket or projectile. A demonstration using a water hose would be useful in showing the shape of the parabola.

Students should be able to write and graph equations of parabolas given their identifying characteristics and identify properties of parabolas from their equations.

The Hyperbola

The general equation for a hyperbola is given as $x^2/a^2 – y^2/b^3 = 1$ *where,* $b^2 = c^2 – a^2$

The hyperbola is the set of all points in a plane such that the absolute value of the difference of the distances (focal radii) from two given points (foci) is constant. Unbound objects traveling in space may be used as an example of the hyperbola.

Unified Conics

The general formula for conic sections is given as: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

Previously we have looked at the specific conic sections. The circle as applied to Ptolemy's universe, ellipses in Kepler's laws, parabolas in missile or rocket flight, and hyperbolas in the path of asteroids and other extraterrestrial objects. We will now look at the generalized equation for conic sections and how we can determine which form will be produced by a specific equation. The following table lists the relationship between *A* and C for each conic section described by the general equation when *B* = 0.

(table available in print form)

Defined in a different manner eccentricity can be expressed in the following table

(table available in print form)

In other values, Mercury (with an eccentricity of 0.2056) holds the title as the largest value among the planets of the Solar System. Prior to the redefinition of its planetary status, the dwarf planet Pluto held this title with an eccentricity of about 0.248. The Moon also holds a notable value at 0.0554.

Most of the solar system's asteroids have eccentricities between 0 and 0.35 with an average value of 0.17. Their comparatively high eccentricities are probably due to the influence of Jupiter and to past collisions.

The eccentricity of comets is most often close to 1. Periodic comets have highly eccentric elliptical orbits, whose eccentricity will be just less than 1; Halley's Comet's elliptical orbit having a value of 0.967. Non-periodic comets follow near-parabolic orbits and thus have eccentricities very close to 1. Examples include Comet Hale-Bopp with a value of 0.995086 and Comet McNaught with a value of 1.000030. As Hale-Bopp's value is less than 1, its orbit is elliptical and so the comet will in fact return (in about 4380AD). Comet McNaught on the other hand has a hyperbolic orbit and so may leave the solar system indefinitely. Planet Neptune's largest moon Triton is believed to be the only astronomical body that has a perfectly circular orbit with an eccentricity of 0. [3]

(image available in print form)

Figure 5. *source:* Wikipedia

Classroom activities

Each of the class activities will correspond to the subject headings above for a total of six lessons. In outline form the lessons are as follows:

1) Discovering Conics (Introduction)
2) The circle
3) The ellipse
4) The parabola
5) The hyperbola
6) Unified conics

First lesson

The first lesson will include a hands on activity where students will be given styrofoam cones along with a piece of piano wire for slicing the cones. The students will be instructed to slice the cones at different angles. The students will describe in writing the various shapes that are formed.

There will also be a demonstration with a sundial. In a darkened room a high intensity lamp will be moved about the gnomon of a sundial. Students will write their observations.

Third lesson

For the lesson on the ellipse the class will begin with discussion of Kepler's laws. The students will then complete the following worksheet:

Worksheet

Work with a partner. Place the piece of paper with the vertical line and horizontal line on the cardboard in the landscape position (so that the horizontal line is longer than the vertical line). Push the pushpin into the center of the paper where the two lines intersect. Take the string loop and place it around the pushpin. On the other end of the loop position your pencil and take the slack out of the string. Trace all the way around.

1. What shape have you drawn?
2. If the horizontal line is the *x* axis and the vertical line is the *y* axis, write the standard equation for this shape.

Now, take the two pushpins and place them on the horizontal line equidistant from the center such that there will still be some slack if the string loop is placed around the pushpins. One partner should secure the pins as the other takes the string loop and places it around the pushpins. Take the pencil, removing slack from the string loop, and trace out a figure.

3. What shape have you drawn?

The points where the pushpins are placed are each called the focus; in the plural, foci. Label them F_1 and F_2 respectively. Now, pick any three points on the figure you have drawn and label each P_1, P_2, and P_3. Using your ruler, measure the distance, as best as you can, from each focus to each point on the figure to complete the table below.

Distance in cm | F_1 | F_2 | Sum $F_1 + F_2$

P_1

P_2

P_3

Using a new sheet of paper create two more ellipses using different points on the horizontal (*x*) axis. Be sure that the foci points are equidistant from the center. Complete the tables below for each.

Distance in cm | F_1 | F_2 | Sum $F_1 + F_2$

P_1

P_2

P_3

Distance in cm | F_1 | F_2 | Sum $F_1 + F_2$

P_1

P_2

P_3

4. The sum listed in the last column of each table is also known as the focal radii. Looking at the sums in the last column, what conjecture are you able to make about the focal radii? (remember, there may be some measurement error).

Now, using the three ellipses you have already drawn, measure the distance along the horizontal (longer) axis to each vertex of the ellipse and write it in the table below in the column labeled major axis. From the tables above also write the focal radii (Sum $F_1 + F_2$) for each of the ellipses.

(table available in print form)

5. Using the data from the table above, what conjecture are you able to make about the relationship between the focal radii and the major axis?

Now, measure along the vertical (shorter) axis from vertex to vertex and write the result below.

(table available in print form)

Assume your ellipses were drawn on the co-ordinate plane with the vertical line as the *y* axis and the horizontal line as the *x* axis in order to answer the following questions.

6. What is the relationship between the *x* co-ordinate of the vertex and the measure of the major axis?

7. What is the relationship of the y co-ordinate of the vertex and the measure of the minor axis?
8. What conjecture can you make about the measure from each of the foci to the vertex at the minor axis?

Use the above data and your conjectures to make generalizations about ellipses in order to answer the following questions. It may also be useful to recall the distance formula.

9. If the length of the major axis of an ellipse centered at the origin is 10 units, what are the x co-ordinates of the vertices?
10. If the length of the minor axis of the same ellipse is 6 units, what are the y co-ordinates of the vertices?
11. What is the distance from each of the foci to the vertex at the minor axis?
12. What will be the x co-ordinates of the foci for this ellipse?
13. For the ellipses we made today we know that one half the measure of the major axis (semi-major axis) will equal the absolute value of the x co-ordinate of the vertex of that axis. What is the quotient of the absolute value of the x co-ordinate of the vertex of the major axis divided by the length of the semi-major axis (i.e., what is the quotient of a number divided by itself)?
14. What is the quotient of the absolute value of the y co-ordinate of the vertex and the measure of the semi-minor axis?

Fourth lesson

Using a garden sprayer a demonstration of a parabolic trajectory will be shown. Students will then answer a series of questions on what factors will alter the trajectory, such as pressure, sprayer height, sprayer angle.

Sixth lesson

The final lesson will include an interactive website where students will answer the following questions: What conic section do you get when 0 e 1? When e > 1? When e = 1? How does the shape of the conic section change as e gets closer to 0? How does the shape of the conic section change as e gets closer to 1? Explain why the eccentricity is equal to the distance from the vertex E to the focus F, divided by the distance from E to point A on the directrix. Drag point P on the ellipse. What happens to the distances between P and the focus and between P and the directrix? What happens to the ratio of these distances? Why is it correct to say that the eccentricity of a given ellipse is constant? Does the eccentricity remain constant as the distances between P and the focus and P and the directrix change? Describe the changes that occur in the graph and in the eccentricity as point E moves closer to the focus and as it moves closer to the directrix. What do you understand better about conic sections as a result of working this problem? [4]

Bibliography

Bennett, J., M. Donahue, N. Schneider, M. Voits, *The Cosmic Perspective* , Lebanon, IN: Addison Wesley. 2000. This is a good teacher resource.

Freedman, R.A and W.J. Kaufmann *The Universe* , New York, NY: W.H. Freeman & Co. 2000. I particularly enjoyed this book as a good introduction to astronomy for myself.

Gauss, Karl Friedrich. *Theory of Motion of the Heavenly Bodies Moving About the Sun in Conic Sections: A Translation of Theoria Motus* , Mineola, NY: Dover Publications. 2004. In addition to borrowing a bit from the title of this work I found the preface to be quite interesting. As it turns out, the significance of this particular work is in the realm statistics, specifically regression analysis.

The University of Chicago School Mathematics Project, *Functions, Statistics, and Trigonometry* , Scott Foresman Addison Wesley. 1998. This is the textbook used for this unit.

Web Sites

http://www.krellinst.org/UCES/archive/resources/conics/ http://www.geom.uiuc.edu/docs/reference/CRC-formulas/node29.html http://www.math.com/tables/algebra/conics.htm http://www.astronomyforbeginners.com/astronomy/gravity/intro. Great for students. http://curious.astro.cornell.edu/question.php?number=191 http://www.answers.com/topic/conic-section http://www.britannica.com/search?query=conic%20section&ct=eb&fuzzy=&iq=10&show=10&start=11

Unified Conics: http://www.keymath.com/x7223.xml. Great interactive applets.

Vector Projection: http://www.keymath.com/x7206.xml

General Conics: http://www.keymath.com/x7227.xml

http://www.mathopenref.com/ellipsesemiaxes.html

http://en.wikipedia.org/wiki/Orbital_eccentricity

Notes

1. New Haven Public Schools. Precalculus Curriculum

2. The University of Chicago School Mathematics Project, *Functions, Statistics, and Trigonometry* , Scott Foresman Addison Wesley. 1998

3. http://en.wikipedia.org/wiki/Orbital_eccentricity

4. http://www.keymath.com/x7223.xml