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Mathematics at the Frontier of Astronomy

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Overview of the Unit

Astronomy has been a subject that has been of interest to the earliest civilizations through the centuries. Over the years different civilizations have contributed to the advancements in astronomy these advances have led to the ongoing exploration of space and thus increased our knowledge of our universe. The explorations and observations in astronomy have also added to the contents of both science and mathematics.

The objectives of this unit are:

1. to introduce students to the study of astronomy
2. to connect topics in the mathematics curriculum to topics in astronomy
3. to provide interdisciplinary activities for students in mathematics.

The curriculum unit is written in three sections. The first section gives an overview of the earliest astronomers and their contribution to the field; the second section consists of topics from astronomy and their mathematical implications; the third section includes lesson plans that could be used to teach the concepts. The sections are designed as mini lessons. The first section of the curriculum unit gives the setting and the background that support the units. In some cases the background pertinent to the unit is included in the lesson plan.

SECTION 1

Early Astronomers

Star gazing has been a part of nearly every culture since the earliest known civilizations. These early men saw almost the same configurations that can be seen in the sky without the powerful telescopes. The ancient star gazers thought that they recognized certain patterns and configurations of stars. They used these patterns and configurations to generate myths and associate them with mystical events. They named these configurations of the stars constellations. [1]

Different civilizations had different interests in astronomy. In the earliest agricultural societies tracking the celestial movements had implications for generating an accurate calendar to keep track of the seasons. The sailors and nomadic tribes used stars to help them navigate large bodies of water and vast areas of desert regions. Knowledge of the skies and its content was used in religion by the early priests to make predications and therefore gave them some control over their congregations. [2]

The Greeks

The Greeks were the first to develop a mechanical model to describe the structure and the operations of the universe. In the fourth century B.C., Plato contributed two attributes to the celestial bodies. He stated that they are spheres attached to spherical shells and that their motions are circular. [3]

Aristotle's view of the universe was that it consists of a set of nested, transparent, crystalline spheres centered on the Earth. There spheres rotates at different speeds and on different axes, and carried the Sun, Moon, stars, and the planets around the Earth producing the spectacular sights seen in the sky. His model consisted of a total of 55 crystal spheres. This model was improved and refined by other astronomers. This model was a complex system. It provided a reasonable approximation of the celestial motions. Its thesis was that the earth was fixed at the center of the universe, and all bodies circled the Earth. The Greeks constructed a model to stimulate what they saw in the sky and used mathematics to explain it. From observational data they concluded that the Earth was round and not flat, and they were able to give a nearly accurate estimate of the size of the Earth. The observation was made mathematically by Eratosthenes when he observed the shadow cast by the sun at the same time in two different cities. This generated the curved Earth model.

Contribution of the Arabs

The geography of the region in which the Arabs lived was suitable for observing the stars. The clear desert sky provided them the opportunity to watch the movement of the stars. After the decline of the civilization of the Greeks the Arabs under Islam had become very enlightened. They preserved and translated the writings of Aristotle, Ptolemy and other Greek philosophers. These works were then stored in their libraries.

In the 9th century the Persian astronomer al Farghani wrote about the motions of the celestial bodies. In the 10th century an observatory was built in Iran, by the astronomer al- Khujandi. Using his data from his observations of the Sun he was able to calculate the obliquity of the ecliptic i.e., the tilt of the Earth's axis relative to the sun. Omar Khayyam revised the existing calendar, and then developed a calendar that was almost as accurate as the one used presently. He used his data to calculate the number of days in a year. He produced a year that had 365.242198558156 days. [4]

The Copernican Revolution

In 1543 the published works of Copernicus " *On the Revolution of the Celestial Spheres* " challenged the concept that the Earth was at the center of the universe. He proposed a new model for the arrangement of the universe. In this system it was proposed that the Sun was at the center of the universe, and that the Earth was a planet and was at the center of the Moon's orbit. He produced a model of this concept. This model assumed circular orbits of each planet as they travel around the sun.

Copernicus realized that Mercury and Venus are always observed near the Sun, he therefore suggested that their orbits must be smaller than the Earth's. These planets were called the *inferior* planets. However Mars, Jupiter and Saturn are seen opposite the Sun, he concluded that these planets must have orbits larger than the Earth's and these planets were called *superior* planets.

Copernicus used the heliocentric model to determine the time that the planets take to complete one orbit. This is called the period. He was able to distinguish between two different periods of each planet. The *synodic period* is the time that elapses between two successive identical sights seen from the Earth. The *sidereal period* is the true orbital period of a planet, that is the time takes the planet to complete one full orbit of the sun.

The *synodic period* is found from observations of the skies where as *sidereal period* can only be found by calculations. A combination of triangles and geometry and the observation from the *synodic period* Copernicus generated the following formula 360/S = 360/Pe-360/Pm, where S = the planet's *synodic period* , P_E = the Earth's sidereal period and P_m = *Mars's sidereal period* . This data was generated from the data of Mars and was then to generalized to 1/S = 1/P1-1/Pe, where

P_E = the sidereal period of the Earth = 1 year

P1 = the sidereal period of the inferior planet

S = the planet's synodic period

The sidereal periods of the superior planets were used to determine the distance from the sun.

Table 1 The distances of planets from the sun [5] .

Planet	Copernicus(AU)	Modern Value (AU)
Mercury	0.38	0.38
Venus	0.72	0.72
Earth	1.00	1.00
Planet	Copernicus(AU)	Modern Value (AU)
Mars	1.52	1.52

Jupiter | 5.22 | 5.30

Saturn | 9.17 | 9.54

1 AU = 1 astronomical Unit. Copernicus used the sidereal periods of the superior planets to help him determine the distance of these planets from the sun.

Brahe and Kepler

Tycho Brache (1546 - 1601).Tycho Brache's observations solidified the heliocentric theory. He attempted to measure the distance of a new star that was seen in 1572. He discovered that the distance of the star was too far to use parallax to calculate its distance. Again in 1577 he attempted to use parallax to measure the distance of the comet, but he also failed. His observations proved that the heavens were not fixed and unchanging. With the support of the Danish king who made him a new observatory he measured the position of the stars and the planets. Even though he never accepted the Copernican [6] model of the universe because he could not prove mathematically the stellar parallax that should have accompanied it he achieved the highest accuracy in measurement without the use of the telescope.

The work of Brahe and Kepler dispelled the concept that the orbits of the planets around the sun were circular, and established from observed data that the orbits were elliptical. From Brahe's data observational data Kepler, a keen mathematician was able to deduce from the data of the planets that their orbits were not circles but were ellipses. From the data Kepler was able to extract three generalities. These became known as Kepler's Laws.

Kepler's Laws

After he established that the orbits were elliptical, he attempted to study the shape and the motion of the planets. He needed a law to describe the shape of the orbits and another to specify the speed of the planets as they moved along their orbits.

1. Kepler's First Law states that the path of each planet is an ellipse with the sun at one focus.

Once Kepler knew the shape of the planets' orbits, He described exactly how they move on the orbits. He found that has the planets travel in the elliptical orbits their speeds changes. These changes were dependent on the distance from the sun. The planets move more rapidly when they are closest to the sun. This point on the orbit is called the perihelion. The planets moves more slowly when they are farthest from the sun. This point is called the aphelion. This law he called *the law of equal areas* .

2. Kepler's second law states that a line from the sun to the planets sweeps out equal areas in equal times.

From Tyco's data he calculated the motion of the planets. He found that there was a relationship between the size of the planets and the time it takes to complete a revolution round the sun. He found that the larger the semimajor axis which determines the distance from the sun, the longer the sidereal period.

3. Kepler's third law states that square of the orbital period of a planet is proportional to the cube of the semi-major axis of the orbit. This law indicates that larger planets are move slower when they revolve around the sun. It quantified the relationship between orbital speed and orbital size and explained the findings of Copernicus. The third law written as an equation is given as $P^2 = a^3$, where, P = planet's sidereal period, (the time it takes the planet to orbit the sun) in years a = planet's semimajor axis, in AU.

Kepler's laws simplified the models that existed. They made it easier to calculate the motions of the planets and they also produced more accurate results. The laws are significant today because they can be applied to calculate the orbits of modern space crafts and to calculate the orbits of other relationships of other stars in their orbits.

Galileo's observations: The Italians Contribution

Galileo's contribution was made from the use of the observations he made with his telescopes. He discovered that Jupiter had four moons, and that those moons continually changed their positions. Galileo's observations about the moons circulating Jupiter led him to question whether or not the sun was being circled by the planets. He communicated these observations to Kepler, who concluded that the moons obeyed his harmonic law.

Galileo was also the first to detect the irregularities of the moon's surface. These observations were contradictory to the earlier concepts that the celestial bodies were perfect spheres. His observations of dark spots on the sun led to the belief that the sun also rotated. He also observed the phases of Venus. This finding solidified Copernicus's model that the planets rotated around the sun.

Galileo also made major contribution to the field of physics [8] . The branch of physics known as mechanic was developed from his work with falling objects. From the observations from falling objects he formulated the law of inertia. This law was in opposition to the concepts of the early Greeks that claimed that an object at rest was in a natural state. Galileo's law of inertia stated that a body at rest is a special condition because that any object once set in motion will continue to move until a force is applied to it.

Kepler's work and the contribution of Galileo laid the foundation for the study of the solar system

Newton's Laws of Motion

Isaac Newton introduced a new approach to the study of the planets and their motions. The previous astronomers deduced their findings from observations. Newton on the other hand started with some general statements that related to the effects of forces on bodies and then showed how Kepler's laws applied. These laws not only describes the effect of the forces on Earth but also were applied to the heavenly bodies.

Sir Isaac Newton's three laws of motion enabled us to understand the motion of the bodies in the solar system. The laws relate the motion of the bodies to the force acting on them. The basic concepts of force and mass were defined.

The First law: The Law of Inertia.

This law of inertia states that a body continues in its present state of motion unless it is acted on by a net external force. This law was opposite to the current belief but states that a body at rest is a special case. In this case the velocity is zero, with no force acting on it to cause it to move. A body will continue in a straight line until a force is applied to it. It was shown that this law was incomplete, because no consideration was given for effects of the force of the universe on that body. Since the body did not exist in a vacuum and the forces of the universe had an effect. It was concluded that to keep the body moving in a straight line for ever, the body must exist with nothing else. The law of inertia enables us to understand how the planets are able to maintain a constant path around the Sun, because of an unbalanced force is applied to it.

Newton's Second Law: The Law of Forces

The Law of forces relates to the change in the velocity of a body to an unbalanced force applied to the body. It states that the net force applied to a body is equal to the mass of the body times the acceleration caused in the body times the acceleration caused in the body by that force. The acceleration is in the same direction as the force. This law expressed as an equation is $F = ma$.

Acceleration measures the change in the velocity of that body. Velocity involves how fast an object is moving and also the direction of the motion. Acceleration is a result of a change in speed, in direction, or a change in both speed and direction. Quantitatively acceleration can be expressed as the amount of change in velocity divided by the interval of time.

Let Delta x = change in speed

Delta t = change in time

Then a = Delta x / Delta t

The law of force was also be used to measure the mass of an object. The mass of a body (the amount of matter) is defined as the resistance of the body to change in motion.

$F = ma$ implies that two bodies have the same mass if the same force applied to the bodies causes the same acceleration.

The Third Law: The Law of Reaction

The law of reaction states that if one body exerts a force upon another body, the second body exerts a force equal and opposite to that force on the first body. This means that if a body exerts a force on another body causing it to move, the first body also moves. This movement is caused by the equal and opposite force exerted on it by the second body. Their acceleration can be calculated using Newton's Law of force. The smaller mass will experience a greater acceleration, while the larger mass will experience less acceleration. This is used to explain the Moon -Earth rotational system. The forces on both the Earth and the moon are the same strength, but the Moon's mass is less than that of the Earth, therefore the Moon does most of the moving.

Newton's Law of Gravitation

Newton explained the concept of gravity as the force that makes an object fall to the earth's surface, and used it to explain the planetary motions. He used the size of the Moon's orbit around the earth and its orbital period to calculate its acceleration.

Working with Kepler's law of planetary motion Newton found the force law required to explain the planetary motions. In this process he developed a new force law called the Law of Universal Gravitation and it is applicable throughout the universe. He also developed a new form of mathematics called calculus. He used this new mathematics to calculate the force of gravity between two bodies given as [9]:

$F = G (m_1 M_2 / d^2)$, where

G = the constant of gravitation

m_1 and M_2 = the masses of attracting bodies

d = the distance between the centers.

The force of gravity is always directed along the line joining the two bodies, therefore this law can be applied to a body on the Earth's surface as well as the Earth itself. The combination of the law of gravitation and the law of force is used to express the acceleration of a falling body at the Earth's surface in terms of the Earth's radius and mass. In the equation $F = ma$ the gravitational force given by the law of universal gravitation as:

$F = G (mM_{earth} / R_{earth})$

$ma = G (mM_{earth} / R_{earth}^2)$, and solving for a we get

$a = GM_{earth} / R_{earth}^2 = g$

The gravitation of earth is denoted by g. This has a value of $9.8\ m/sec^2$. To test the gravitational law Newton used the data from the moon into the formula. It proved that the gravitational force decreases as the square distance between the bodies.

Connections between Kepler's Laws of Planetary Motion and Newton's Laws of Motion and Gravitation

Newton's laws of motion and gravitation can be to generalize Kepler's laws. Newton was able to predict a number of phenomena which were verified by observations.

By the law of reaction:

1. If two bodies are attracted to each other, both must be accelerated, therefore both are in motion.
2. Since the forces of gravity always lie along the line joining the two bodies, the common point about which they move must on that line.
3. The opposite forces of the bodies are equal therefore the acceleration of the bodies will be inversely related to their masses.
4. The larger body of mass will show the least motion, while the smaller body of mass will show the greater motion.
5. The common point around which the bodies move is called the center of mass. The product of the mass first body (m_1) and its distance from the center of mass (r_2) equals the product of the mass of the second body (m_2) and its distance from the center of mass (r_2). This is given as $m_1 r_1 = m_2 r_2$

When Newton's calculus is applied to his laws of motion and gravity Kepler's first law of planetary motion can be extended.

1. When two bodies under the force of mutual gravitation move on similar orbits about a common center of mass their orbits will always be of three kinds of curves: ellipses, parabolas, or hyperbolas. If a body has an elliptical orbit, then it repeats its path indefinitely. For both the parabolic orbits and the hyperbolic orbits since they represent open curves, the body in those orbits will have only one gravitational encounter with another body.

2. Kepler's law of equal areas represents a general law when Newton's theory is applied. The application reads; the line joining two bodies will sweep out equal areas in equal units of time. This law holds for any force law as long as there is a directional property, and, will apply whether the orbit is elliptical, parabolic or hyperbolic. The law implies that the bodies will move much faster in their orbits when the distance is smaller, and much slower when there are at a greater distance.

3. Kepler's harmonic law applies to elliptical or circular orbits and works only when one object orbits another. The relative orbit is made up of the sum of two similar individual orbits. Newton's introduce a new term the sum of masses and is given by the equation $(m_1 + m_2) P^2 = a^3$ where

m_1 and m_2 = the masses of the bodies expressed as solar masses

P = the orbital period in years

a = the semi- major axis of the relative orbits expressed in astronomical units. The sum of the masses expression is very useful in astronomy. It can be used to calculate the masses of stars.

Newton recognized that Kepler's laws were applicable for the case of two isolated interacting bodies, but failed in the case of the solar system where the attraction of the planets on each other causes deviations in their orbit. Kepler's laws would only work for all planets if they were the only an individual planet and the sun were present. The small forces exerted by the planets are minute when compared to the force of the sun. These small interplanetary forces are called perturbing forces, and the deviations caused on the orbits of the planets are called perturbations.

The Origin of the Solar System

There are two major theories that have been advanced to explain the origins of the solar system. Any of the theory that is accepted need to explain or satisfy three leading questions. 1. The differences in composition between the terrestrial planets and the Jovian planets. 2. The fact that all the planets orbit the sun in the same direction and all their orbits are in nearly the same plane and 3. The terrestrial planets orbit close to the sun while the Jovian planets orbit far from the sun. The two theories that have been advanced by astronomers are the nebular theory, and the Core Accretion hypothesis.

The Core Accretion hypothesis

This theory has been studied for a long time by astronomers. It states that planets are formed by the accretion of planetesimals and gases from the solar nebula [10]. Under conditions of low temperatures and low pressure a substance cannot remain as liquid, its condensation temperature determine whether it will end up either as a solid or a gas. The gas cloud from which the solar system formed had initial temperature near 50 Kelvin so these substances could have existed in the solid form.

This condition changed as the central part of the solar nebula heated up to form the proto-sun. As the proto-sun collapsed and heated up, the temperature of the inner solar system rose and that of the outer planet remained cool. This explains why the inner planets are rocky and the outer planets remain icy.

The planets became larger by the accretion onto smaller bodies. As a planet grows, it tends to get even larger because its gravity causes more material to accrete on it. The planets captured smaller bodies, creating more space between each planet.

The Nebular Theory

The nebular theory that has been accepted was the theory proposed by Immanuel Kant in 1755. His proposal was not tested, but with today's data the theory seems to describe the true origin of the solar system.

Observations have shown that regions of diffuse matter between the stars have dark globules, huge clouds of cold gas and dust. These large clouds contract under their own gravity. As they shrink, their matter is compressed, thus becoming hotter. As they contract they rotate faster and faster and their shape become distorted. A distinct equatorial bulge is developed as a result of the contraction. The rotation and contraction continue until the material in the disc acquires orbital velocity about the sun. This concentration of gas continues to revolve around the sun in a circular motion. After the concentration of the disc stops, the heating of the disc ceases. The sun prevents the inner regions from cooling off rapidly, while the outer regions become colder by radiation of heat.

The radiation of the solar system depends on the chemistry of the cooling gases. There are three facts that are consistent with respect to the gases that form the solar system

1. The gases consist mostly of hydrogen and helium. The existing temperature in the disc causes the hydrogen and helium to remain as gasses.
2. The rest of the matter in the disc consists of some light atoms, such as oxygen, carbon and nitrogen. These can form solids at very low temperature.
3. The less common, heavier atoms such as magnesium, aluminum, silicon, calcium, and iron are capable of forming solids at higher temperature.

The formation of the Planets.

In the cold outer part of the disc the temperature remained very low. Low enough for the molecules to form into ice. In the inner part of the disc, the Sun's heat prevents it from becoming cold, but remains hot so that only heavier atoms like iron and molecules of materials can condense. These particles orbit the Sun, and as they collide with each other and pick up other particles. They become large and therefore attract the particles that come near to them. The largest particles attract most of the matter, thus growing large and becoming planets.

In the inner regions where it is too hot for lighter elements to condense, the rocky and metallic planets like Mercury, Venus, Earth and Mars were formed [11] . Mercury was formed in the hottest region. It has the highest iron content, and the highest density. The elements that form the rocky planets are relatively scarce and are very heavy these planets are very small and dense. The planets formed in the colder regions are more massive and less dense. Jupiter and Saturn are so massive that their gravity captured large amounts of hydrogen and helium gasses. It should be noted that neither of the hypothesis presented explain planetary systems other than our own. Hence other hypotheses are being examined.

The classification of the Solar System

The solar system consists of other bodies beside the planets and the Sun. There are the satellites of the planets; the asteroids and the comets. In addition the planets can be differentiated by their composition (the terrestrial planets and the Jovian planets), by their location (the inner planets, and the outer planets), and by their sizes (the larger planets, and the smaller).

The Structure and Composition of the Comets

Comets are small objects composed of ices and dust. They spend million of years in the outer regions of the solar system in their frozen state before being perturbed into smaller orbits around the Sun. The particles that form the dust do not evaporate. They are released and orbit the Sun with great eccentricity, when they are near the Sun they vaporize, and form a luminous head and a long tail. The spectrum of a comet is the reflection of the sunlight by the dust particles.

The head of the comet expands as it nears the Sun. The head is surrounded by a huge hydrogen cloud. The hydrogen comes from water molecules and hydrocarbon molecules broken up by the sunlight. The tail of the comets always points away from the Sun. This is caused by the radiation pressure of the solar winds [12] . Therefore the tail of the comet precedes it in parts of the orbit. When the comet gets close to the Sun more material is fed into the tail. The tail gets larger as the comet passes the Sun and diminishes as it moves away from the Sun.

The brightness of the comets increases as they get close to the Sun, because more material evaporate and the sunlight excites the gasses and reflect the dust become more intense. As a result the brightest comets are those that get closest to the Sun. Each time the comet passes the Sun they lose material by evaporation. In some cases the effect is so strong that that the comet can split in pieces. After a number of solar passages the comet will evaporate. Most comets are found to have elliptical orbits and approach the sun with great speeds. These planets on elliptical orbits are bound to the Sun and belong to the solar system. As the comets enter the inner region of the solar system can have their orbits affected by the planets. Some of the planets speed up and their orbits are converted into a hyperbolic orbit. These comets make their last trip about the Sun and then depart from the solar system.

The orbits of comets unlike the planets are not restricted to a flat system but can have any inclination. The orbits of comets have semi-major axes of almost 25, 000 AU. This proves that they are at very far distances from the Sun.

SECTION 2

The Mathematical Connections

Mathematics has provided the tools to enable astronomers to quantify their observations. This section will discuss a selection of the concepts and their application to mathematics.

Escape Velocity

When a ball is thrown in the air, it rises then it falls back down. If more velocity is applied it would go higher before falling back down. The question is how fast must a ball be thrown so that it does not return that is fall back down. The smallest velocity that will prevent the ball from returning is called escape velocity. Escape velocity is defined as the minimum seed at which a projectile at a planet's surface has to be launched in order to permanently leave the planet.

The task is that we want to give the ball enough kinetic energy so that it will go away to a distance where the Earth's gravitational force is zero therefore using up all the kinetic energy. $E_{initial} + E_{input} = E$ final $+ E_{output}$ gives the equation for the conservation of energy. The $E_{initial}$ consists of the kinetic and potential energy of the ball when it was released from the earth's surface. After the ball is released there is no energy input or energy output. The ball arrives at infinity with zero kinetic energy and zero potential energy [13].

The equation $E_{initial} + E_{input} = E_{final} + E_{output}$ can be written as

$$\tfrac{1}{2} m v^2 - (GMm/R) + 0 = 0 + 0$$

$$V_e = (2Gm/R)^{1/2}$$

The relation is applicable for any spherical, symmetrical gravitating body of mass *M* and radius *R* . The escape velocity *V e* depends on the mass and size of the large gravitating body, (e.g) the Earth not the ball. We can therefore calculate the escape velocity for the earth and other planets. We will need to know the mass and the radius of the bodies.

Orbital Motion

From the orbital motion of two gravitating bodies information about the mass of the bodies can be calculated. If the two masses orbiting under the influence of the same gravitational force, they will have orbits about a common point. This point is called the center of mass.

The orbits will always be of one of three kind of curve an ellipse, parabolic, or hyperbolic.

The center of mass

The center of mass is a point directly between the two masses. A straight line can be drawn to connect both masses. Let *m* represents the smaller mass and *M* represents the larger mass, then the distance of *M* from the center of mass represented by *R* and distance of *m* from the center of mass is represented by *r* . This distance can be expressed as

MR = mr.

For very large objects like the planets the center of mass can be located at the geometric center of the body. For example in the Earth- Moon orbital system, the center of mass is inside the Earth but not at the geometrical center. Both the Earth and the Moon orbit about a common center of mass, but the orbit of the Earth is so small because of its mass when compared to the Moon, it is said that the Moon orbits the Earth. The center of mass of the Earth - Moon system is located in the center of the Earth. The Earth and the Moon orbit around a common center of mass. For two bodies of considerable differences in masses the center of mass is closer to the heavier body. The concept of the center of mass can be extended to systems of a number of mass points.

The larger mass *M* has a gravitational attraction on the smaller mass *m* . Therefore

Newton's second law for gravitational for masses can be applied. The law states that the acceleration of an object is proportional to the net outside force acting on the object.

In an equation this is expressed as *F = ma* . Given that *F* = GMm/r^2 then GMm/(r+R)^2 = m(v^2/r). When *v* = (2 Pi ^2 * r)/ T. Replacing *v* with *2Pi * r* we have GM / (r+R)^2 = (4 Pi ^2 * r)/T^2 this equation becomes T^2 = (4 Pi ^2 * d^3) / [G(m+M)] *d =(r + R)* and *T = period* .

For a small mass orbiting a much larger mass, the equation can be applied to estimate the mass of the heavier body if the orbital data of the lighter mass is known.

Problem: Estimate the mass of the sun given the following information

Let M = Mass of the Sun

Let m = Mass of the Earth

d = 150,000,000 km [14]

T = 1 year

Converting to mks and substituting in the equation, the mass of the Sun

$M = 2.00 \times 10^{30}$ *kg.* The mass of the Sun is used as the unit for expressing the mass of other stars and other large bodies. The symbol used for the mass of the Sun is M.

Gravity

Gravitational attraction is a fundamental property of matter. The gravitational force is extremely weak. It is the weakest of the four known forces in nature. The other forces are the strong and weak force, nuclear forces, and the electromagnetic force. Gravitational force is of importance when one of the bodies has a very large mass, as large as a planet or a star, or any other large celestial body. Sir Isaac Newton described mathematically this universal force of gravity. He found that gravity extended above the domain of Earth to other bodies.

The measurement for gravity ' *G* ' was determined by a British scientist Henry Cavendish in 1797. For his experiment two small spherical masses of the same weigh m in kilograms were attached to the ends of al light rod. This system was suspended by a fine wire so that the system is balanced. Two larger masses M were placed at the opposite ends of the smaller masses. The gravitational attraction between the masses twisted the wire. The elastic property of the wire was known, so it was possible to measure the force of attraction between the masses m and M. Substituting the found measurement in the equation $F_{gravity} = GmM/r^2$. The value 6.67×10^{11} $N.m^2/kg^2$ is used for *G* .

Newton's second law for freely falling bodies of mass M near the Earth's surface and the value of *G* and GmM/R^2 = mg, was used to calculate the mass of the Earth. Where *M* is the mass of the Earth and *R* the radius of the Earth the calculations gives [(6.67 * 10^-11 * Nm^2 / kg^2)*M] / (6.38 x 10^6 *m)^2 = 9.8m/s^2. This gives $M = 5.98 \times 10^{24}$ kg for the mass of the Earth.

Cosmic Distances

The distance of a star from sun is very large. The approximate distance of the closest star to the Earth is almost 250,000 times the distance of the Earth to the Sun. Because of these great distances no direct measurement of these distances can be applied. To measure stellar distance astronomers have used geometrical method to determine these distances. This process is called parallax measurement [15] .

Parallax is defined as the angular or apparent change in the position of an object due to the change in the position of the observer. A distant star will appear to change its position as the Earth moves on its orbit around the Sun. The angle between the two lines of sight is called the parallax angle. The size of the parallax angle is dependent on the distance between the observation point and the distance being observed. The parallax angle of a star determines the distance to the star. The further away the star is the smaller the parallax angle.

It is important to have consistency when comparing the parallax angle of different stars thus determining their distances. Astronomers agree to use 1 AU [16] as the same base line for the two observations that will be used for measuring the parallax. Thus the parallax of a star is the parallax angle that will be measured from two points separated by exactly one astronomical unit and with the line between the points perpendicular to the direction of the star. The value determined is called the trigonometric parallax.

To measure a star's parallax astronomers measure a star's distance using two points of view.

1. When the Earth is at the point P_1 in its orbit, set the telescope to receive light from the star to be measured and the distant sky beyond. The star should be in the center of the view.
2. Six months later when the Earth is at point P_2 with the telescope to receive light from the same star under the same conditions, turn the telescope through twice the angle of the previous setting to place the star in center the of view.
3. Since the distance from the Sun to the Earth is known the measurement of the angle generates a right triangle. Trigonometry can be used to measure the side of the triangle that corresponds to the distance from the Sun to the star.

The direction from the Earth to the star changes as the planet orbits the sun, and the nearby star appears to move back and forth against the background of more distant stars. The parallax of a star apparent position shifts as the Earth moves from one side of its orbit to the other. The larger the parallax " *p* " the smaller the distance ' *d* ' to the star.

Let P_1 = the first observation

Let P_2 = the second observation (six months later)

Let p = the parallax (angle between the line of sight and the perpendicular line from the sun)

Let d = the distance to the stars.

The distance d is measured in parsecs. The unit of measure called the parsec written as pc is defined to be the distance for which the parallax is exactly one second. *If* $p = 1$ arcsecond, by definition $d = 1$ *pc* .

The following relationship can be written between p (the parallax) and d the distance.

$d = 1/p$ [17] where d = distance of a star in pc and p = parallax angle of that star in arc seconds. Other relationships: *1 pc = 3.26 light years (ly) =* 3.09×10^{13} *km,*

= 206,265 AU . (0ne second of arc = one thirty six hundredth of a degree)

Trigonometric parallax is used only for measuring the distance of stars in the solar neighborhood. For more remote stars where *p* is very tiny other methods for measuring distances are required.

Stellar Magnitudes

The system used by astronomers to measure the brightness of stars is based on a classification developed by Ptolemy in the second century AD. He compiled a catalogue of several hundred stars and then grouped stars into six categories according to how bright they appeared to the naked eyes. He classified the brightest stars as first magnitude and the dimmest stars as sixth magnitude.

Today astronomers recognize that the first magnitude stars are about one hundred times as bright as the sixth magnitude stars. To establish a more precise scale a star of magnitude 1 is defined as 100 times as bright as a star of magnitude six. A star of n^{th} magnitude will be exactly *x* times as bright as a star of magnitude $n + 1$. To find an approximation for x by using the fact that the first magnitude star is exactly x times as bright as the sixth magnitude star, and by definition a first magnitude star is 100 times as bright as the sixth magnitude star [18] . Therefore

$x^5 = 100$

$x = 2.51188...$

Each magnitude of brightness corresponds by a factor of approximately 2.5. Therefore a star of magnitude n is about 2.5 times as bright as a star of magnitude $n+1$.

The apparent brightness (B_{app}) of a star depends not only on the amount of light it emits (intrinsic brightness) but how far away it is from Earth. If the stars could be moved so that they are all the same distance from the Earth the star's B_{app} (apparent brightness) would give a good measure of the intrinsic or absolute brightness (B_{abs}). This would allow us to compare the brightness of the stars, or to say which star is the brightest.

If we know the distance to a star, then using the inverse square law for intensity we can determine how bright the star would be if it were moved to some other distance. If this were done for all stars whose distance is known, we can choose some standard distance ($d_{standard}$) and calculate how bright each star would be if moved from its actual distance (d_{actual}) to this standard distance. The inverse square law gives the equations for apparent brightness and absolute brightness as

$B_{abs} = C / d_{standard}^2$ and $B_{app} = C / d_{actual}^2$ where C represent intrinsic properties of the star that do not depend on where the star is located. If equations are divided by equation 1, then it gives $B_{abs} / B_{app} = (d_{actual} / d_{standard})^2$

At its actual distance a star has magnitude m . This is called the apparent magnitude. The magnitude that a star would have at the standard distance is denoted by M . This called the absolute magnitude. If the star's actual distance is greater than the standard distance ($d_{actual} d_{standa_r d}$), then $m > M$. This means that the star is brighter at the standard distance. The difference in brightness can be calculated using the magnitude difference factor 2.5, so at the standard distance it would be $2.5^{m - M}$ times as bright as a the actual distance. Therefore $(d_{actual} / d_{standard})^2 = 2.5^{m - M}$. In this equation it is assumed that the observed brightness of a star depends only on its intrinsic brightness and on how far away the star is located. The effect of the other particles between the observer and the star that would diminish the observed brightness is ignored. Astronomers use the standard distance to be 10 pc (parsec is magnitude in light years). The equation can therefore be written as $(d/10pc)^2 = 2.5^{m - M}$, where d is the actual distance of the star.

SECTION 3

Lesson Plans: Connecting the unit to the curriculum

Lesson Plan I

Topic: Measurement in Astronomy

Purpose

To investigate the units of measurement used in astronomy

To express/ convert distances in astronomical units

Background Information:

The vastness of space requires us to describe distances and other measurements that are either very large or very minute. These sizes are beyond human comprehension. Therefore for us to grasp this vastness Astronomers developed units of measure that are suitable to describe the situations. Interstellar distance is measured in light years or parsecs, and the mass of the sum, or other stars expressed as *M* . Scientific notation is used to express very large and very small units. This makes easier to write, read and calculate.

When expressing the measure of human sizes the either SI units of conventional units of ft and inches are used. To express very small units micrometer are used.

1 micrometer = 1um = 10^{-6} *m*

1 nanometer = 1 nm = 10^{-9} *m*

To express mass Astronomers use grams, kilogram or solar masses, the following scales are used:

1kg = 1000g

1 M = 1.99×10^{30} *kg*

Astronomers also differentiate between mass and weight. Mass is defined as the measure of material in an object and weight is defined as the pull or force that gravity acts on a body. It is usually expressed in pounds or new tons (*1 Newton = 0.225 pounds*).

Speed is measured in SI units as meter / seconds. It is also expressed km/s as well as mi/h. The following is the conversion table:

1km/s = 10^{3} *m/s*

1km/s = 2237m /h

1mp/h = 0.447m/s

1 mp/h = 1.47 ft/s

To measure the vast distances across the solar system astronomers use a unit of length called the astronomical unit (AU).

1 AU = 1.496×10^{8} *km = 92.96 million miles*

1 AU represents the average distance between the Earth and the Sun.

To measure the distance to the stars astronomers use two different units

a) Light year (ly): This is the distance that light travels in empty space. The speed of light in empty space has a value of 3.00×10^{5} km/s on 1.86×10^{5} mph/ s. Therefore one light year can be expressed as *1Ly =* 9.46×10^{12} *km = 63,240 AU.*
b) Parsec (pc): This unit of length is 3.26 light years.

1 pc = 3.09×10^{13} *km*

= 3.26 ly

1 kiloparsec = 1kpc = 1000 pc = 10^{3} *pc*

1 Megaparsec = 1 Mpc = 1,000, 000 pc = 10^{6} *pc*

Classroom Problems

1. The average distance from Earth to the Sun is 1.496×10^{8} km. Express this distance in a) light years b) in parsecs. Express in powers of ten.
2. The universe is estimated to be about 13 billion years express this in seconds.

Lesson Plan II

Topic: Using regression to explain Kepler's Law that there is a relationship between the planets' orbits and their distance from the sun.

Materials:

Graph paper, Graphing calculator

Objectives: Students will be able to

a) Create a scatter plot from the given data
b) Identify the variables
c) Write the equation that can be used to model the data

Table 2: Planets distance from the sun and their periods

(table available in print form)

Classroom Problems/ Activities.

1. Draw a scatter plot to show the relationship between the distance from the Sun and the period.

2. Describe the relationship

3. Perform the following transformations of the data. Then explain the graphs

   i) Distance v/s log (period)
   ii) Log (distance) v/s log (period)

4. Use the result in (ii) to generate a power model.

Activity II

The table given shows the relationship between the angle of parallax and the distance of a star in light years.

Table 3: Relationship of Angle to Parallax

Angle of Parallax | Distance in Ly

0.5 sec | 6.5 ly

0.25 sec | 13.0 ly

0.1 sec | 32.6 ly

Use the equation *d* = 1/p where *d* = distance and *p* = parallax angle of a star in arcseconds to add 4 pieces to the table.

   a) Use the table to explain the relationship between the angle of parallax and the distance in light years.
   b) Generate an equation that could be used to model the relationship between angle of Parallax and the distance to a star.
   c) Draw a graph that could be used to model this data.

Lesson Plan III

Topic: Kepler's contribution to the study of astronomy.

Purpose:

To apply Kepler's laws to a problem situation

To investigate the properties of Ellipses

Make the connection to Newton's Laws

Background Information

Kepler's first law states that the path, or orbit of a planet around the sun is an ellipse, the position of the sun being at a focus of the ellipse.

Definition of an ellipse: A set of all points *(x, y)* in a plane. The sum of whose distance from the foci is a constant.

A line through the foci intersects the ellipse at two points called the vertices. The chord joining the vertices is the major axis, and its midpoint is the center. The chord perpendicular to the major axis at the center is the minor axis.

Kepler's second law a line joining the planets and the sun sweeps out equal areas during equal intervals of time.

Kepler's third law "the squares of the orbital periods of planets are directly proportional to the cubes of the semi - major axis of the orbits.

$P^2 = a^3$

P = the orbital period in years

a = semi major axis of orbit

Strategy

Review the meaning of sidereal period for a given planet.

Review the parts of the ellipse

Review the meaning of astronomical units

Discuss Kepler's third law

Worked Sample problem

Significant Task/ Problems

1. A piece of solar debris in said to be orbiting the sun. It takes 9 years to complete the orbit. Determine the semimajor axis of the orbit
2. The average distance from Mercury to the Sun is 0.39 AU. Determine the sidereal period of Mercury.

Newton's form of Kepler's third law

Kepler's law can only applied to objects in orbit, but Newton's form can be applied to any situation in which two bodies of different masses orbit each other. The equation is given as:

$P^2 = [4\,Pi^2 / G(m_1 + m_2)] * a^3$, where

P = sidereal period of orbits in seconds

a = semimajor axis of orbits, in meters

m_1 = mass of first object, in kilograms

m_2 = mass of second object in kilogram

G = universal constant of gravitation = $6.67 \times 10^{-11}\,N.m^2/kg^2$

Significant Task

Two Hubble telescope on its mission to outer space discovered a new planet. This planet and its moon orbit each other as in the case of the Earth and the Moon. You are an intern in the NASA Space Agency and you are asked to evaluate the data sent back and report the masses of these two planets. From the data you calculated the following information

The orbital distance is 1.888 days and planets are 294,700 Km apart.

Lesson Plan IV

Topic: Escape Velocity

Purpose: The students will be able to apply the equation of escape velocity to a problem situation.

To have students research the concept of Escape velocity.

Material

Calculator

Significant Task

You are taking part in a science fair and decided to design a spacecraft or a satellite to orbit one of the planets. The rubric for the evaluation specifically states that the escape velocity for the planet you choose to orbit.

Prepare a presentation explaining your design and the escape velocity necessary for the planet of your choice.

Lesson Plan V

Topic: Comparing the Brightness of stars

Objectives: Students will be able to

a) Determine the brightness of a star
b) Differentiate between apparent brightness and absolute brightness
c) To compare the brightness of two stars

Background Information

The system that modern astronomers use to compare the brightness of a star is based on the classification developed by Hipparcus and formalized by Ptolemy. Stars were placed in six categories according to how bright they appeared when they are seen in the sky without a telescope. The brightest stars were given first magnitude and the dimmest stars were assigned magnitude six. It was recognized that a first magnitude star was 100 times brighter than a sixth magnitude star. To make the computation easier the magnitude difference of 5 corresponds exactly to a factor of 100.The equation $x^5 = 100$ establishes the size of a magnitude with $x = 2.512$. Therefore each magnitude difference in brightness corresponds to a factor of about 2.5. A star of magnitude n is 2.5 times brighter than a star of magnitude $n+1$.

Sample Problems

1. The star that appears brightest in the sky is Sirius. It has an apparent magnitude of - 1.46. The other bright star is Regulus with an apparent magnitude of 1.35. How many times as bright as Regulus is Sirius?
2. Two stars are observed in the sky. Star A has an apparent magnitude of 16.9. The other star B has an apparent brightness of 5.0. Which is the brightest and by how much?

Reading List for Teachers

De Vorki, David H. (1986). Practical Astronomy, published for the National Air and Space Museum by the Smithsonian Institute Press. Washington, D.C.

Encernaz, T., Bibring, J.P. & Blanc, M. (1987). Astronomy and Astrophysics. Springer - Verlag. New York.

This text provides the major content of astronomy and provide sample problems.

Freedman, Roger A., Kaufmann, William J. (2000). Universe. W. H. freeman & Company. New York.

This is the text book recommended for the seminar. It covers the material extensively and provide the mathematical implications in the section called tool for the trade.

Holliday, Keith (1998). Introductory Astronomy, John Wiley and Sons, New York.

Protheroe, W. M., Capriotti, E.R., & Newson, G. H. (1981). Exploring the Universe. Second Edition. Charles E. Merril publishing Company, Columbus, Ohio.

This is an older text book written for non math undergraduate students in astronomy.

Seaborn, James, (1998). Understanding the Universe: An Introduction to Physics and astrophysics, Springer- Verlag, New York.

This text provides a good integration of the mathematics and physics and astronomy. It is an excellent resource for problems.

Notes

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1. W.M Protheroe, E.R Capriotti & G.H. Newsom, Exploring the Universe (Columbus: Charles E. Merrrill Publishing Company 1981) 2.

2. James Seaborn , Understanding the Universe: An Introduction to Physics and Astrophysics (New York: Springer- Verlag, 1998) 2

3. James B Seaborn, Understanding the Universe: an introduction to physics and astrophysics (New York: springer- Verlag) 2.

4. http:en.wikipedia.org/wiki/history of astronomy May 16 2007

5. W. M. Protheroe et al, Exploring the Universe (Columbus: Charles E Merrill 1981) 19

6. Roger a. Freedman, William J., Kaufman, Universe (New York: Freedman & Company, 2005) 69

7. 7 W.M Protheroe et al, Exploring the Universe (Columbus: Charles E Merrill 1981) 20

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10. Sarbani Basu lecture notes, Frontiers of Astronomy Seminar 2007.

11. W.M Protheroe et., al, Exploring the Universe (Columbus: Charles E Merrill 1981) 206

12. T. Encrenaz, J.P. Bibring & M. Blanc, The Solar System (New York: Springer- Verlag 1987) 18

13. James B Seaborn, Understanding the Universe: an introduction to physics and astrophysics (New York: springer- Verlag) 96.

14. James B Seaborn, Understanding the Universe: an introduction to physics and astrophysics (New York: springer- Verlag) 106.

15. Roger a. Freedman, William J., Kaufman, Universe (New York: Freedman & Company, 2005) 410

16. James B., Seaborn, Understanding the Universe: an introduction to physics and astrophysics (New York: springer- Verlag) 198.

17. Roger A. Freedman, William J., Kaufman, Universe (New York: Freedman & Company, 2005) 410

18. James B Seaborn, Understanding the Universe: an introduction to physics and astrophysics (New York: springer- Verlag) 201.

19. Roger A. Freedman, William J., Kaufman, Universe (New York: Freedman & Company, 2005) 68

Appendix: Connecting to the Mathematics Standards

The NCTM standard in mathematics outlined that students should be given the opportunity to connect mathematics to other subjects areas.

Standard 8: Communication. This standard states that students should be given the opportunity to:

a) Organize and consolidate their understanding of mathematics through communication
b) Communicate their mathematical thinking coherently and clearly
c) Analyze and evaluate the mathematical thinking and strategies to others
d) Use the language of mathematics to express mathematical ideas.

Standard 9: Connections This standard states that

a) students should be given the opportunity to recognize and use connections among mathematical ideas
b) Recognize and use mathematics in contexts outside of mathematics
c) Understand how mathematical ideas build on each other to produce a coherent whole.