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Gravity: A Relatively Heavy Subject

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Introduction

The development of mathematics, from its very beginnings, has been about problem solving. Whether it has been commerce, trade or navigation we have always used early mathematical explorations of our world for very practical purposes.

Although the title suggests that Einstein's theory of relativity may be incorporated into the unit, it will only be mentioned in passing. One of the main themes of the unit will be the role of mathematical modeling in describing our world. Despite more advanced theories, Sir Isaac Newton's model is still widely used to this day.

The study of standard gravity, planetary motion, and the motion of tides, in addition to the quest for knowledge about the world around us, was required in order for human beings to be able to better navigate, and map, the vast world around them. It is in this context, a practical applications approach, in which I would like to develop this unit.

Metropolitan Business Academy is an inter-district magnet high school in New Haven. The students have primarily chosen to pursue careers in business. Historically these students have done better on the verbal portion of standardized tests such as the CAPT. It is the intent of this unit to draw upon these superior verbal skills to teach about the Fundamental Theorem of Calculus and periodic functions.

Rather than focus on the abstract and procedural nature of the subject, the unit will attempt to put a human face on what is a very human endeavor. The unit will use the story of Isaac Newton, and his discoveries, to illustrate the scientific process and the role of mathematics in that process. Additionally, the unit will emphasize the problem solving and practical aspects of the whole enterprise. By having students "discover" ways to solve these problems in a historical context we hope to facilitate a deeper and better understanding.

The seminar will allow me access to source materials which will be invaluable in the preparation of this curriculum unit. The materials will be used to develop a curriculum which is relevant to students of business in the practical application of solving real world problems.

The New Haven calculus curriculum covers the Fundamental Theorem of Calculus and the Precalculus curriculum covers periodic functions. The unit specifically directs students to solve real world problems. As with many - if not most - mathematical discoveries, Newton's work was developed in conjunction with solving real world problems. In this case it is the study of the universal force of gravity and the effect upon planetary motion and tides on Earth. The unit will be used to teach students in this problem solving, real world, context.

The running theme through the unit will be the contribution of Newton's discoveries of universal gravitation as it specifically relates to standard gravity, planetary motion and the role of gravity's influence upon the ocean tides. Although there are more modern theories of gravity, calculus was developed concurrent with Newton's theories of motion and gravity. As such it is more appropriate for study in an introductory calculus course.

Newton's work in explaining and predicting ocean tides will be used in a Precalculus course demonstrating the practical application of periodic functions. By making and recording observations students will learn something of the scientific method. Students will also come to better understand the role of mathematics in that process. The power of mathematical models will be demonstrated.

When Newton's *Principia* was published, it was thought by many to be impenetrable. Others thought it was a step backward by invoking some sort of demon or supernatural force. Scientific thinking (particularly Descartes) had quelled some of the importance people attributed to invisible spirits, but Newton invoked an invisible force called gravity, a force which ruled the apple, the Moon, and the Earth, and which caused the tides. Newton could only maintain that it was not occult. The essence of gravity was not something that he or anyone else understood, but it was demonstrated by mathematics. "It is enough," he wrote, "that gravity really exists and acts according to the laws that we have set forth and is sufficient to explain all the motions of the heavenly bodies and of our sea." For Newton, the mathematical model was sufficient.

The physicist Richard Feynman liked to tell a story about how when he was a little kid, he asked his father, "Why do things fall?" As an adult, he praised his father for answering, "Nobody knows why things fall. It's a deep mystery, and the smartest people in the world don't know the basic reason for it." Feynman liked his father's answer, because his father realized that simply giving a name to something didn't mean that you understood it. The radical thing about Galileo's and Newton's approach to science was that they concentrated first on describing mathematically what really did happen, rather than spending a lot of time on untestable speculation such as Aristotle's statement that "Things fall because they are trying to reach their natural place in contact with the earth."

The fundamental theorem of calculus, in physical terms, states the relationship between acceleration, velocity, and position (or displacement). Given the function of one it is possible to calculate the others. The unit will use falling objects and other motions to demonstrate the principle.

Moon phases and the tides may be modeled with periodic functions. Students will collect and use data to derive periodic functions to model this behavior.

Planetary Motion

Nearly everyone has heard about Newton and an apple. But few people seem to know the story behind it. Technically, there is no actual documentation for this story, so it might contain exaggerations. But it is relatively well accepted as having happened.

Prior to this incident, Newton had invented the Calculus, and with it had mathematically proven that an "inverse square law" dependence, such as gravitation on distance, must act as though all the mass of an object (the Earth) is at the exact center of the Earth.

Newton was trying to think of some way of experimentally confirming what he had already calculated, that inverse square dependence. He was sitting out in a field, looking at the Moon in the sky overhead. He believed that the Moon was orbiting the Earth because of the gravitation of the Earth. He believed that the Moon would normally have gone straight off into space, but the Earth's gravitation caused it to "constantly fall" toward the Earth, making its path curved rather than straight. But he hadn't thought of any way to experimentally prove that.

By his time, science had fairly accurately calculated the radius of the Earth, just under 4,000 miles (6,400 km). It was also known that the Moon orbited the Earth at an average distance of just under 240,000 miles (384,000 km, about 60 times as far from the center of the Earth as he was. These things were known.

When an apple fell from a tree near him, it suddenly dawned on him that the same Earth's gravitation that must be curving the Moon's path must also have made that apple accelerate toward the Earth in its fall.

His calculations had shown that the acceleration should not depend at all on the size or mass of the object. So, if that apple was out at the distance of the Moon, it should have the same acceleration as the Moon does, and would therefore also orbit the Earth. He knew that an apple falls at "the acceleration due to gravity", 32 feet per second per second, what we call **g** . And that in the first second, that apple would fall very close to 16.1 feet (193") toward the Earth.

Then, if that apple was moved to a place 60 times as far away from the center of the Earth, and gravitation actually did depend on an inverse square relationship, then the apple out there should fall 1/3600th as far as it did from the tree. So he multiplied 16.1 feet by 1/3600 and got an expected falling distance in one second to be 0.0535 inch.

That meant that the Moon must "fall" 0.0535" toward the Earth in a second (from an otherwise straight line. This is a small curvature (less than 1/16" over the 3,300 feet that the Moon moves every second!). But it turns out that it is still pretty easy to confirm. If you draw a really big circle that represents the orbit of the Moon, and then look at a small part of that circle, the part that the Moon moves through in one second, then simple geometry can determine that small curvature. (circle, chord, radius, etc.)

Interestingly, in this very simple calculation, the brilliant Newton apparently made a multiplication error regarding the radius of the Earth in inches! With this wrong value, there was no agreement in the results. Newton set aside this whole subject for six years! Around then, a new calculation of the radius of the Earth had been made (by Picard). Newton decided to try the calculation again, and he did it right this time, and the result was 0.0534", a virtually perfect match. The inverse square law of gravitation was therefore proven. Also proven was the fact that the mass of the object, whether apple or Moon, did not affect the acceleration results.

As to this last statement, Newton later calculated that there actually IS a tiny effect due to the mass. But it is an extremely tiny effect, for any practical sized objects, because the Earth is so big and massive. There is also a tiny effect due to the differential gravitational effect of the Sun, which very slightly reduces the actual value for the Moon, which even explains that 0.0001" discrepancy.

History of Tidal Theory

The earliest references to tidal differences were made by the Greek astronomer and explorer *Pytheas* around 330 B.C. On a voyage to the British Isles he observed the great ocean tides there. The tidal differences were much greater than those found in his native Mediterranean. He discovered that there was some sort of relationship with the moon. Pytheas not only discovered that there were two high tides per lunar day, but also that the amplitude depended on the phases of the moon.

The Mediterranean did not provide a good observatory for tidal theory because the amplitude (magnitude) of the tides was not as pronounced there. About 150 B.C. *Seleukos* made observations in the Red Sea. He found that the two tides per day had unequal amplitudes when the Moon was far away from the equator.

The Greek geographer Strabo quoted the early Greek scientist, Poseidonios in a book written around the year 23 A.D. Poseidonios made studies of the tides of the Atlantic coast of Spain around 100 B.C. Quoted from Strabo's book is the oldest existing text describing the tidal phenomenon of diurnal inequality.

When the moon rises above the horizon to the extent of a zodiacal sign (30°), the sea begins to swell, and perceptibly invades the land until the moon is in the meridian; but when the heavenly body has begun to decline, the sea retreats again, little by little; then invades the land again until the moon reaches the meridian below the earth; then retreats until the moon, moving round towards her risings, is a sign distant from the horizon … The flux and reflux become greatest about the time of the conjunction (new moon), and then diminish until the half moon; and, again, they increase until the full moon and diminish again until the waning half moon. If the moon is in the equinoctial signs (zero declination), the behavior of the tides is regular, but, in the solstitial signs (maximum declination), irregular, in respect both to amount and to speed, while in each of the other signs, the relation is in proportion to the nearness of the moon's approach. [1]

The phenomenon of the diurnal inequality is illustrated in figure 1:

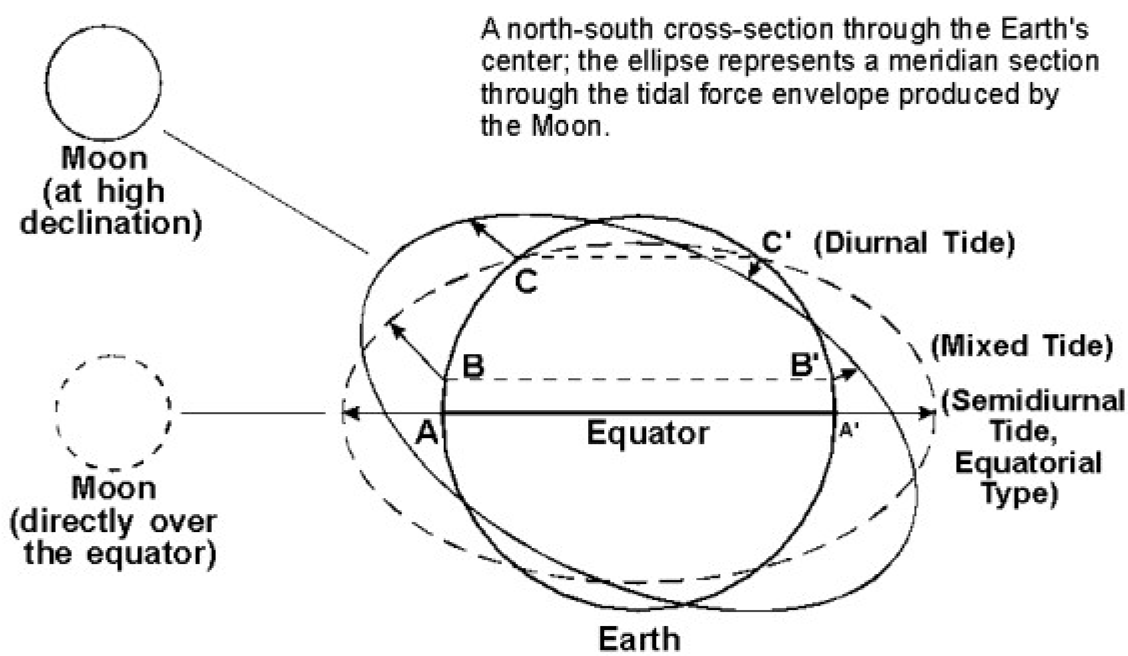

Figure: 1. (source: http://co-ops.nos.noaa.gov/restles4.html)

More than two thousand years ago the ancient Greeks, as with many other phenomena, made astute observations about the world around them. As with many of their other observations they did not fully understand or explain the cause. It would take another 1600 years before the tides' motions would be more fully understood. In the meantime, there were several unsuccessful attempts at explaining the various phenomena.

An English monk in the 8 th century discovered the phase lag of the ocean tides, noticing that each port along the British coast had its own tidal phase. Although Bede the Venerable's observations were accurate, he attributed the ebbing tide to the Moon "blowing" on the water and flowed again when the Moon moved a bit.

Ekman attributes the first scientific attempt to explain the tidal phenomenon to an Arabian scientist in the 13 th century. It is claimed that the sun and the moon heat the waters, thereby making them expand. In Zakariya al-Qazwini's own words:

> As to the rising of the waters, it is supposed that when the sun acts on them it rarefies them, and they expand and seek a space ampler than that wherein they were before, and the one part repels the other in the five directions, eastwards, westwards, southwards, northwards, and upwards [2] .

The hypothesis failed to explain why the moon, not the sun, played the leading role. The whole idea that the sun and moon played a role at all was sometimes difficult to grasp. According to one idea, the tides were caused by the great whirlpool known as the Maelstrom off the coast of Northern Norway. We know today that there is a connection between the whirlpool and the tides, but it is the tide that causes the whirlpool and not the other way around.

With the recent rediscovery of America, in the mid 16 th century it was suggested by an Italian scientist that in addition to the effect of the moon, the sea water was oscillating between the coasts of America and Europe. Apparently this type of resonance had been observed in some of the larger local lakes.

Moving into the 17 th century, Johannes Kepler was convinced that the tides depended on the moon and the sun. He believed that there was some attractive force, something like magnetism. Galileo was surprised that Kepler "became interested in the action of the moon on the water, and in other occult phenomena, and similar childishness" [3] . Galileo himself believed the Copernican theory positing that the tides were produced by the combined effects of the earth's rotation around its axis and its orbital motion around the sun. These motions would set the water on Earth into oscillations observed as tides. He used a thought experiment where the water in a container may be made to oscillate by acceleration and slowing of the container or pan. He concludes that the same must hold true for the vast oceans.

Rene Descartes, slightly after Galileo, had his own ideas. The Cartesian vortex will also be discussed as an explanation for the planetary motions. Both the moon and the earth, in this theory, were surrounded by a large vortex. The pressure exerted by the vortex of the Moon on the Earth's surface caused the tides to rise. Unfortunately, the theory incorrectly predicts a low tide when in reality there is a high tide.

The debate continued through the 17 th century and became quite confusing. What sort of force could cause the oceans to behave in such a way? If the Sun and the Moon were not exerting their influence, through some occult force or otherwise, how could one explain the observations?

An explanation came later in the 17 th century with Isaac Newton's Principia. Universal gravitation was not only causing the apple to fall to the Earth. It was holding the planets in orbit, and also causing the tides because of the varying distances from the Sun and the Moon. In Newton's words:

> But let the body S come to act upon it (the globe), and by its unequable attraction the water will receive this new motion. For there will be a stronger attraction upon that part of the water that is nearest to the body, and a weaker upon that part which is more remote.

Newton's theory was able to explain the three fundamental properties of the tides: the main period of 12 lunar hours, the dependence of the amplitude on the lunar phases, and the diurnal inequality. Figure 1 illustrated the diurnal inequality. Figure 2 illustrates the effects of the lunar phases:

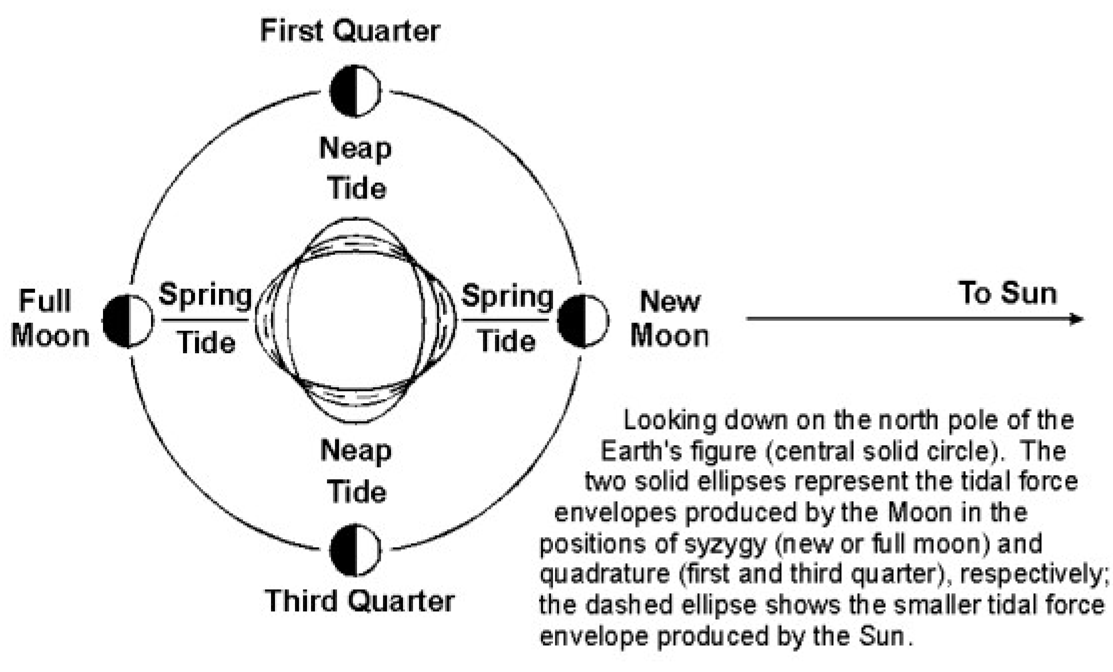

Figure: 2. (source: http://tidesandcurrents.noaa.gov/restles3.html)

While Newton's theories were pivotal in providing the foundation for the development of a mathematical treatment of tides, they were by no means the end of the story. Later mathematicians built upon Newton's epoch-making discoveries much in the same manner that Newton stood on the shoulders of giants in making his discoveries. Such is the nature of scientific discovery. The lunar constituent which produces the twice daily tides is illustrated in figure 3.

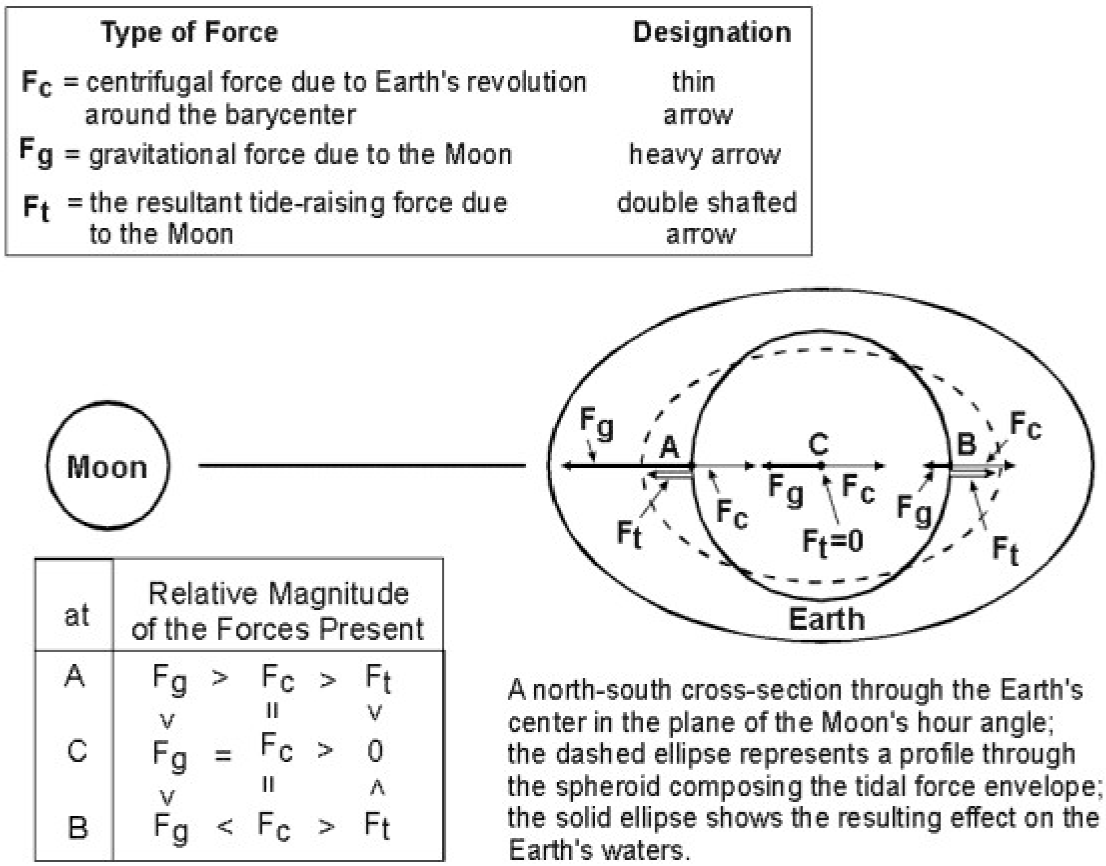

Figure: 3. (source: http://tidesandcurrents.noaa.gov/restles3.html)

It was not until the second half of the 18 th century that a mathematical formula was fully developed by Pierre de Laplace. The tidal potential was described by Laplace in the following manner:

The three preceding terms give rise to three different types of oscillations. The periods of the oscillations of the first type are very long; they are independent of the motion of the earth, and depend only on the motion of the celestial body L in its orbit. The periods of the oscillations of the second type depend mainly on the rotational motion t of the earth; they are approximately one day. Finally, the periods of the oscillations of the third type depend mainly on the angle 2t; they are about half a day [4]. The constituent causing the semi-diurnal tide, centrifugal force, is further illustrated in figure 4:

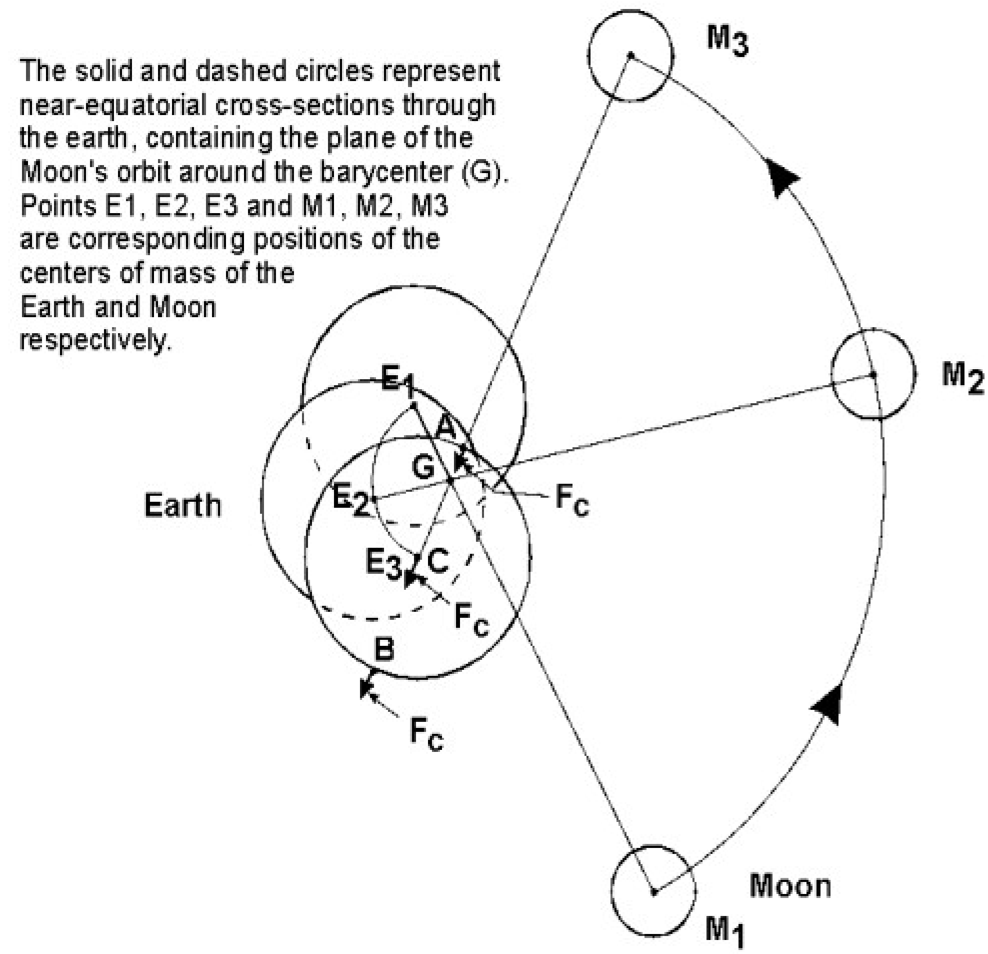

Figure: 4. (source: http://tidesandcurrents.noaa.gov/restles3.html)

The importance of Laplace’s equations should be emphasized. In addition to providing a mathematical model, he was the first to treat ocean tides as a problem of water in motion instead of water in equilibrium. His equations describing ocean tides could not be solved in practice until the invention of the computer.

For purposes of this unit we will deconstruct Laplace’s tidal formula to teach students about periodic functions. Each of the functions will be treated in a discrete manner. If time permits, the students may be introduced to the composite functions. For a more advanced course this unit could be expanded to include Fourrier series in addition to the equations found in this unit. The discrete functions are shown in figure 5:

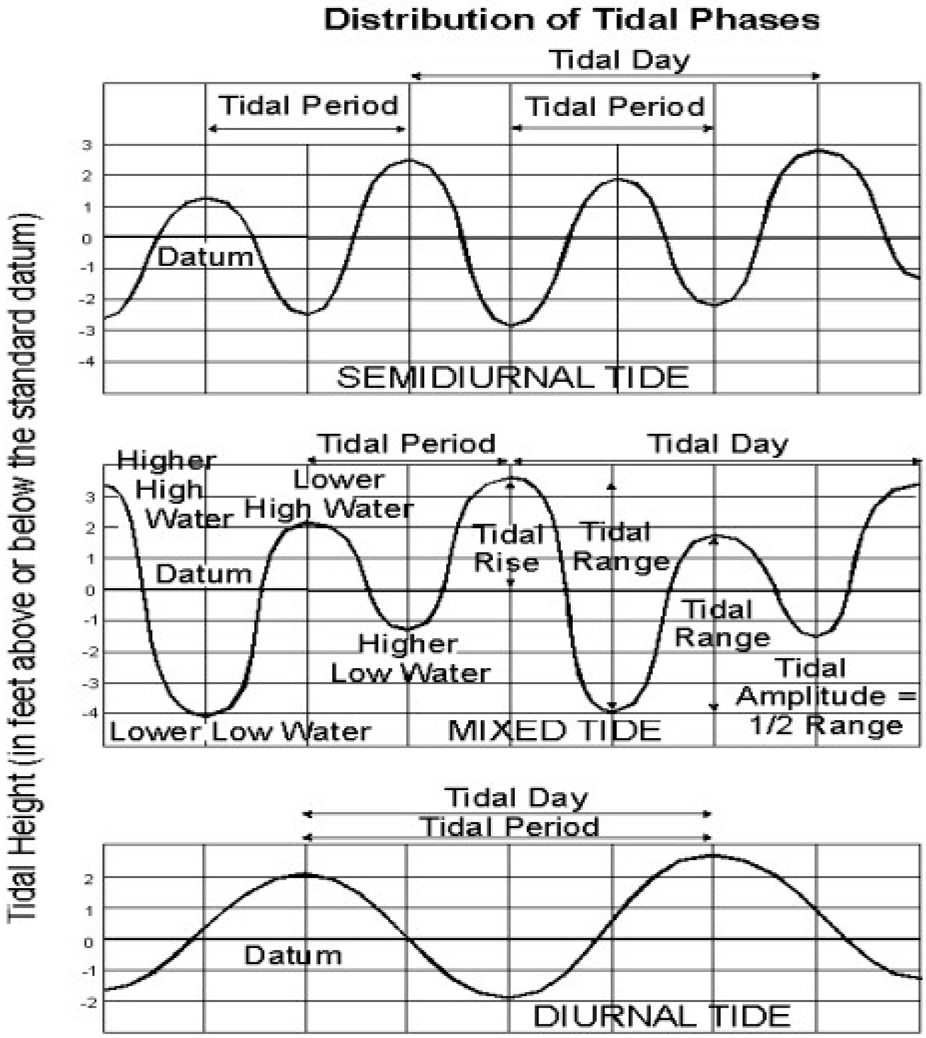

Figure: 5. (source: http://co-ops.nos.noaa.gov/restles4.html)

The primary tidal constituents (ie, the discrete functions), along with their conventional designations are listed below. This table will be useful in understanding the discrete periodic functions which describe the tides.

The following are among the *major tidal constituents* contributing to the astronomical tide:

M_2 - Principal lunar semidiurnal constituent (speed: 28.984 degrees per mean solar hour)

S_2 - Principal solar semidiurnal constituent (speed: 30.000 degrees per mean solar hour)

N_2 - Larger Lunar elliptic semidiurnal constituent (speed: 28.440 degrees per mean solar hour)

K_1 - Luni-solar declinational diurnal constituent (speed: 15.041 degrees per mean solar hour)

O_1 - Lunar declinational diurnal constituent (speed: 13.943 degrees per mean solar hour)

M_4 - First overtide of M_2 constituent (speed: 2 x M_2 speed)

M_6 - Second overtide of M_2 constituent (speed: 3 x M_2 speed)

S_4 - First overtide of S_2 constituent (speed: 2 x S_2 speed)

MS_4 - A compound tide of M_2 and S_2 (speed: M_2 + S_2 speed)

To better facilitate an understanding there may be a demonstration or a directed activity of a tide machine. Students should have some understanding of the trigonometric functions as circular functions. My students have used JAVA applets such as the ones found at http://www.ies.co.jp/math/java/trig/index.html.

Lord Kelvin invented a tide machine to handle complex calculations. It utilized a series of wheels and pulleys to sum the various functions. A demonstration of the machine using a JAVA applet may be seen at: http://www.ams.org/featurecolumn/archive/tidesIII3.html

The United States Coast and Geodetic Survey tide- predicting machine No. 2 was designed by Rollin A. Harris and E.G. Fischer and constructed in the instrument shop of the U.S. Coast and Geodetic Survey. It was completed in 1910 and replaced the Ferrel Tide-Predicting Machine in 1912. The machine summed 37 constituents and was capable of tracing a curve graphically depicting the results. [5]

Lesson 1. Timing the Tides

Tide tables, commonly seen in newspapers and on television in coastal areas, show that comparable local high and low ocean tides occur almost one hour later from one day to the next. The motion of the Moon as it revolves about the Earth largely accounts for this time lag. The following activity investigates the timing of the tides by demonstrating the effect of the Moon's orbital motion on the time that elapses between comparable local tides.

Objectives

After completing this activity, you should be able to:

- Describe how the times of high and low tide change from one day to the next.
- Describe why the times of high and low tide change from one day to the next

Lesson

1. Examine the Tide Time Diagram below. The diagram (not drawn to scale) shows Earth at the center as seen from far above Earth's North Pole. The counterclockwise rotation of Earth and the direction of light arriving from the distant sun are shown. Earth time is marked in one hour intervals along the circumference of the planet. Since Earth rotates 360 degrees in 24 hours, each hour a fixed point on Earth rotates (5) (10) (15) degrees.
2. The large circular ring in the diagram shows the daily positions of the Moon relative to Earth

during one lunar month lasting from one new Moon phase to the next new Moon. (The 29.5-day lunar month has been rounded to 30 days for the purposes of this activity.) Every day, the moon's position advances about (10) (12) (15) degrees along the circle representing the 30-day month.

3. The lower Tidal Bulge/Moon Diagram, viewed from the same perspective as the upper drawing, is of the Moon and Earth with the depth of the ocean greatly exaggerated. It shows the theoretical locations of the ocean's two dominant tidal bulges. One always faces the Moon (where lunar gravitation is strongest) and the other always faces directly away (where lunar gravitation is weakest). Make a tracing or photocopy of the lower diagram on a clear plastic sheet. Place this directly over the upper diagram so that the center points of the diagrams coincide. Use a sharp pencil to hold the two together at their centers. Twist the overlay so the Moon progresses from one daily position to the next. The moon is advancing in the (same) (opposite) direction as Earth's rotation.

4. Place the Moon at its Day 1 position. A point on Earth rotating through the center of the bulge facing the Moon would experience a high tide at this time. The time, found by reading the time on Earth clock indicated by the "Solar Time" arrow pointing at the high tide bulge, would be about (12:50) (1:30) (2:50) p.m.

5. Advance the Moon to its day 2 position. On this day, a point on Earth rotating through the same bulge would experience a high tide at about (12:50) (1:40) (2:30) p.m.

6. Comparing the time of the Day 2 high tide to the Day 1 high tide, the time of the high tide is about (1) (2) (3) hour(s) later than the day before.

7. Just as it takes more than one hour for the minute hand of a clock to make two successive passes of the advancing hour hand, it takes (less than 24) (24) (more than 24) hours for a point on Earth to pass through and catch up with the same advancing tidal bulge. That is why comparable local high and low tides occur later from one day to the next.

The time lag investigated in this activity is typically less than an hour, but how much less? To make an estimate, determine from the diagrams the times of comparable high tide on Day 10 and Day 20. From this information, find how many minutes later the tide occurred on Day 20 than on Day 10, and divide by 10. According to these calculations, the daily time lag rounds off to about (40) (50) (55) minutes.

Lesson 2. Writing an equation to predict the moon phase

Objective

Students will use data to derive an equation to determine moon phase as the percentage of the moon illuminated on any given day.

Procedure

Using data from the U.S. Naval Observatory students will use a graphing calculator such as a TI-83 to make tables of data. Students will plot the data in a graph. From the data students will run a regression to write a sinusoidal equation.

Go to http://aa.usno.navy.mil/cgi-bin/aa_moonill2.pl to obtain data. Assign a different month of data for each student to input. Have the students put in data for every other day, such as first, third, fifth day of each month. L1 should contain the day and L2 should contain the percentage illumination of the moon.

Have students use STAT PLOT to plot a graph of their data. Have students run a TRIG regression of the data. Have students insert the regression equation into the " *y* equals" screen in order to graph the equation. Have students graph the equation.

As students are moving through each of the preceding steps a student, or the instructor, should also be using the TI-presenter. Once the graphs are successfully displayed there should be a discussion centered on the following questions:

> What is different about the graphs and their equations?
> What is the same about the graphs and their equations?
> What is the amplitude of each equation?
> What is the frequency of each equation?
> Why is there a phase shift?

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Notes

1. Ekman, 586
2. Ekman, 586
3. Ekman,587
4. Ekman,590
5. http://www.co-ops.nos.noaa.gov/predmach.html