Curriculum Units by Fellows of the Yale-New Haven Teachers Institute
2009 Volume IV: How We Learn about the Brain

Using Mathematics to Understand the Brain and Describing the Brain to Understand Mathematics

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Mathematics is a powerful tool for solving problems in the world around us. Using very abstract models we are able to describe and predict the sometimes very complex behaviors of people, markets, diseases and physical objects. It is often difficult for students of mathematics to grasp some of these abstract concepts without concrete examples. In particular, it can be a challenge to motivate students without showing some relevance to their own lives. I would like to capitalize on the students' natural curiosity about their own brains to motivate them to learn mathematics.

As a teacher of mathematics at Metropolitan Business Academy (MBA) in New Haven I have used real examples from the seminar. They include comparing the reaction time of a giraffe and a mouse. What is the relation between the number of neurons and brain diameter? How much louder is a jet taking off than a vacuum cleaner? Why do some musical notes sound pleasant while others do not? Relevant mathematical models, and their representations, will be used in answering these questions.

In addition to business careers, many students at MBA will pursue careers in the health professions. Additionally, everyone is curious how the brain works, especially as it pertains to them. The students will use examples taken from the seminar to create mathematical models and then represent the data and their models. These relevant examples will help motivate my students to understand this very abstract subject matter.

The New Haven curriculum for Algebra II and Precalculus includes units on the family of functions. Consequently the curriculum unit developed here could be adopted for either course, although this unit will be devoted to the Precalculus curriculum in order to include logarithm and periodic functions.

Students coming into Algebra II, in particular, as well as Precalculus often do not have a grasp on the properties of various functions and the type of change modeled by each. Furthermore, given a mathematical model, or equation, there frequently is difficulty in displaying data and mathematical models for precise, effective, quick analysis. This unit will develop the students ability to select an appropriate function in order to best represent the data.

The unit will reinforce the notion of data, mathematical models, and graphs as representations of change. The uniquely mathematical perspective of change is the rate of change. In mathematical terms rate of change may be depicted as slope in a graph. By working with data and models, and then making visual representations of that change, it is intended that students will have a better understanding of this concept. Additional skills in rates, ratios and proportions will also be bolstered in depicting and analyzing the data.

Each member in the family of functions will be developed using data from the seminar. The functions included will be linear functions, power functions, quadratic functions, polynomial functions, exponential functions, logarithm functions and periodic functions.

The linear function in slope intercept form is, y = mx + b where m represents slope and b is the y intercept. This function will be used to compare the time it takes for nerve conduction to travel from the foot to the brain of the giraffe and the mouse.

The power function, in function notation, takes the form $f(x) = x^a$, where a is a constant real number. This function will be used to show the relationship between total number of neurons and brain diameter. The quadratic function, a specific type of polynomial function, might also model the same sort of change as above, but takes the form $f(x) = ax^2 + bx + c$, where a, b, c are constant real numbers. The polynomial function utilizes a combination of operations on variables and constants with non-negative whole number exponents. Of these, the third degree polynomial will probably prove most useful.

The logarithm function generally may be stated in any base. The natural base of e is used in many growth functions while this unit will utilize base 10. The conventional function notation is f(x) = ln(x) for natural base and f(x) = Log(x) for base 10. This function, in base 10, will be utilized in measuring sound levels in decibels.

Finally, periodic functions will be demonstrated. These are the trigonometric functions sine, cosine, and tangent. For modeling purposes the sine function will mostly be utilized in the form f(x) = a*sin(b*x + c)+d , where a is amplitude, b is the frequency, which determines period, c determines phase shift, and d determines vertical shift. This function will be used to determine the frequency of sound and the activation of the auditory system.

Using concrete and interesting data the intent is to have students better understand rate of change, how best to model and represent that change by choosing appropriate mathematical models and graphs, and apply skills involving rate, ratio, and proportion. Students will also be introduced to music theory.

Linear functions

The typical nerve cell consists of a body and two kinds of branches. Dendrites receive input from other nerve cells and are usually short. Axons by contrast are often very long and may conduct impulses very rapidly. The axon terminals are positioned near an adjacent nerve or a muscle. Nerve impulses pass from the axons of one nerve to the next nerve or muscle.

Surrounding the axons is an insulating coat of material called myelin. A nerve may be a single axon or bundle of axons. Nerve impulses may move electrically in both directions, but from skin or to muscle it is normally in one direction. (Stewart)

In order to determine conduction velocity stimulating electrodes are placed beneath the skin close to a bundle of nerves. Small recording electrodes are placed on the skin over the muscles being tested. Compound action potential is the sum of firing of many axons within the nerve.

For this unit idealized data will be used to study the reaction times of various animals, as well as humans. Reaction time is nerve conduction from the foot to the brain and back to the foot. The idealized universal conduction velocity will be 100 meters per second. The students will be given data such as the distance from the extremities to the cortex. For example, from the giraffe's toes to his brain is approximately 5 meters. From a mouse's toe to his brain is approximately 0.1 meters.

More detailed procedures and sample data are provided in the lesson plan that follows this section. Students will make tables and from the tables make graphs. Students will then write an equation from the data.

In addition to reviewing basic algebra concepts, students will also be reviewing the use of decimals. This section will also reinforce use of the metric system.

An example of linear equations and their graphs are presented in Figure 1.

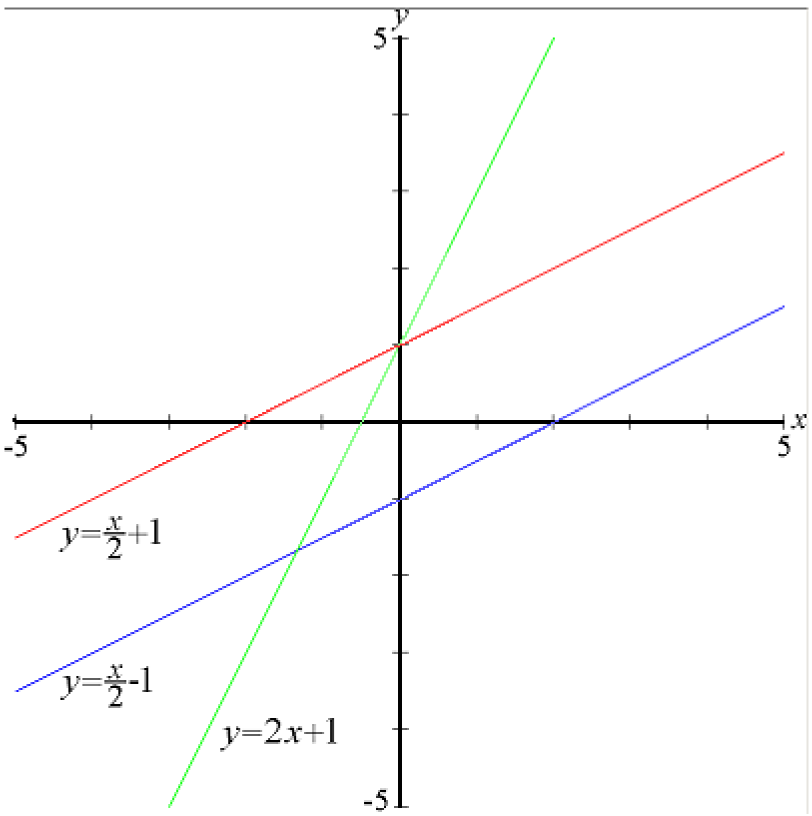

Figure 1. source: Wikimedia

Power function

The power function, in function notation, takes the form $f(x) = x^a$, where a is a constant real number. This function will be used to show the relationship between total number of neurons and brain diameter.

The diameter of a neuron ranges in size from 4 microns to 100 microns. (http://faculty.washington.edu/chudler/facts.html#neuron) Using the equation for the volume of a sphere, $V = (4/3)*\Pi*r^3$, and an average neuron size of 50 microns (1 micron = one millionth of a meter) , students will calculate brains size based upon 100,000 neurons up to 100,000,000,000 neurons. Figure 2 are graphs of power functions.

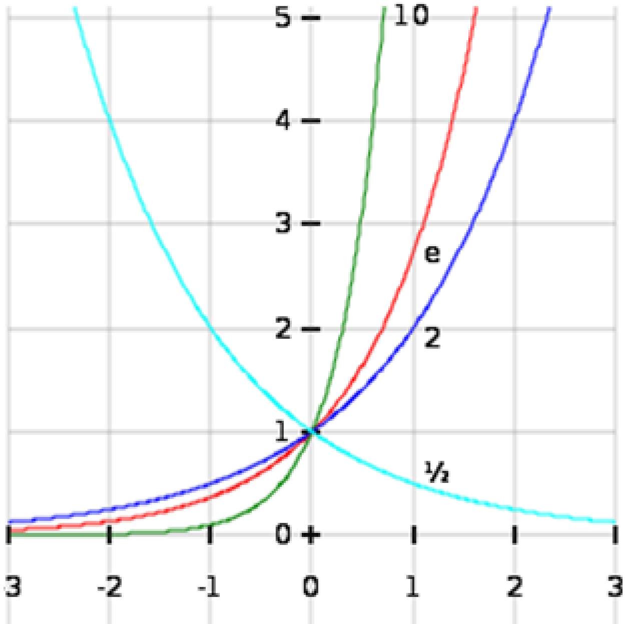

Figure 2. source: Wikimedia

Logarithm function

The logarithmic function, in base 10, f(x) = Log(x), will be utilized in measuring sound levels in decibels. We humans are equipped with sensitive ears capable of detecting sound waves of quite low intensity. The difference in magnitude between what we call the auditory threshold, the softest noise audible to a normal ear, and the loudest audible noise is huge. For example, the magnitude of difference between the auditory threshold and the threshold of pain is some 10,000,000,000,000,000 (ie., ten to the thirteenth power). Logarithms, like scientific notation, are useful when trying to represent or communicate such large magnitudes.

Since the decibel (dB) scale is one of proportion between the baseline, auditory threshold, and an observed or recorded phenomenon the actual unit of measure is not important as the ratio will remain the same. The ratio can be expressed in the following equation: $dB = 10*\log_{10}(P_1 / P_0)$ where is the baseline, or auditory threshold, and is the observed or recorded sound level. While the magnitude of difference between, say a whisper, and normal conversation may appear to be a magnitude of two or three it is actually much larger. An engaging exercise for students might be having them guess the magnitude of difference of some common sounds and then compare with what the actual dB level is.

Below are OSHA permissible noise levels. For example, a normal conversation or TV would be 60 dB, a jack hammer about 100 dB, a major highway also about 100 dB, and a jet taking off 150 dB (http://en.wikipedia.org/wiki/Sound_pressure).

TABLE G-16 – PERMISSIBLE NOISE EXPOSURES

Duration per day, hours	Sound level dBA slow response
8	90
6	92
4	95
3	97
2	100
1 1/2	102
1	105
1/2	110
1/4 or less	115

Footnote(1) When the daily noise exposure is composed of two or more periods of noise exposure of different levels, their combined effect should be considered, rather than the individual effect of each. If the sum of the following fractions: C(1)/T(1) + C(2)/T(2) C(n)/T(n) exceeds unity, then, the mixed exposure should be considered to exceed the limit value. Cn indicates the total time of exposure at a specified noise level, and Tn indicates the total time of exposure permitted at that level. Exposure to impulsive or impact noise should not exceed 140 dB peak sound pressure level.

Another application of logarithms is calculating gestation periods. For example, if we know that a cell will divide approximately each week creating two cells then we can calculate how long it will take for an embryo to fully develop.

As an example we can use the development of the human brain. The newborn has about one hundred billion neurons. The formula to represent this growth pattern is $N = 2^x$. If N equals one hundred billion we can then solve, using the logarithm base 2 as follows: $x = \log_2 100{,}000{,}000{,}000$. Using a calculator we find that x is approximately 36.54 weeks.

Other examples may also be used. The graph of $\log_{10}(x)$ is shown in figure 3.

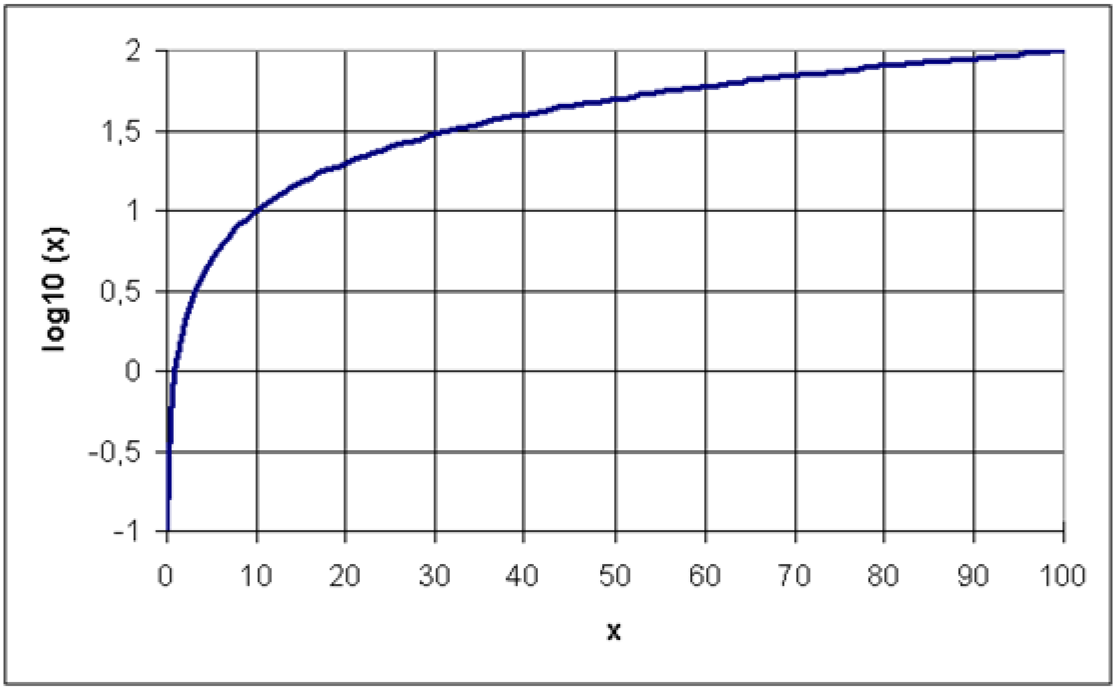

Figure 3. source: Wikimedia

Sine function

Harmony is a function of the relationship between the pitches of different tones either in sequence or sounded at the same time. Within these tonal contexts the pitches set up expectations as to what will come next. The skillful composer arranges these progressions to either meet or violate the listener's expectations for artistic purposes. (Levitin, 2006, p.17).

Jonah Lehrer (2006) suggests that Stravinsky understood, ahead of the science confirming it, that these expectations that we have are learned. If the listener's expectations are not met the piece may be considered disagreeable or discordant. In Rites of Spring Stravinsky violates many of the then current expectations and the work was initially considered discordant. Eventually the piece became accepted and well liked.

Daniel Levitin explains in his PBS show, The Music Instinct, that every object has the capacity to vibrate and make sound. Music is organized sound. When we hear music we are literally being touched and moved. The sound waves hit our eardrums and move the fluid in the cochlea against the hair cells which are laid out from low to high frequency. This is transmitted through the brain stem to the auditory cortex which is laid out in pitch order. It was previously thought that there was a music center in the brain. It is now believed that there are many parts of the brain involved, much like a neural orchestra. Different parts of the brain are affected by the different musical elements of pitch, timbre, tempo, harmony, and melody.

One apparent universal element of music worldwide is the octave. There also appears to be a certain predilection for consonance as opposed to dissonance. The elements of lullabies seem to be universal in the type of consonance that they have.

Dissonance and consonance may be graphed using sine waves. The sine wave related to a musical pitch has the following form, where A is the amplitude of the sound (or the volume, measured in decibels) and B is the frequency of the note (measured in Hz):

$f(x) = A\sin(Bx)$. See figure 4 where the frequency B determines T, the period of the function. Period is calculated as (2Π / frequency).

Students will be directed to graph a variety of chord patterns and determine, by listening and looking at their graphs, which combinations are dissonant or consonant.

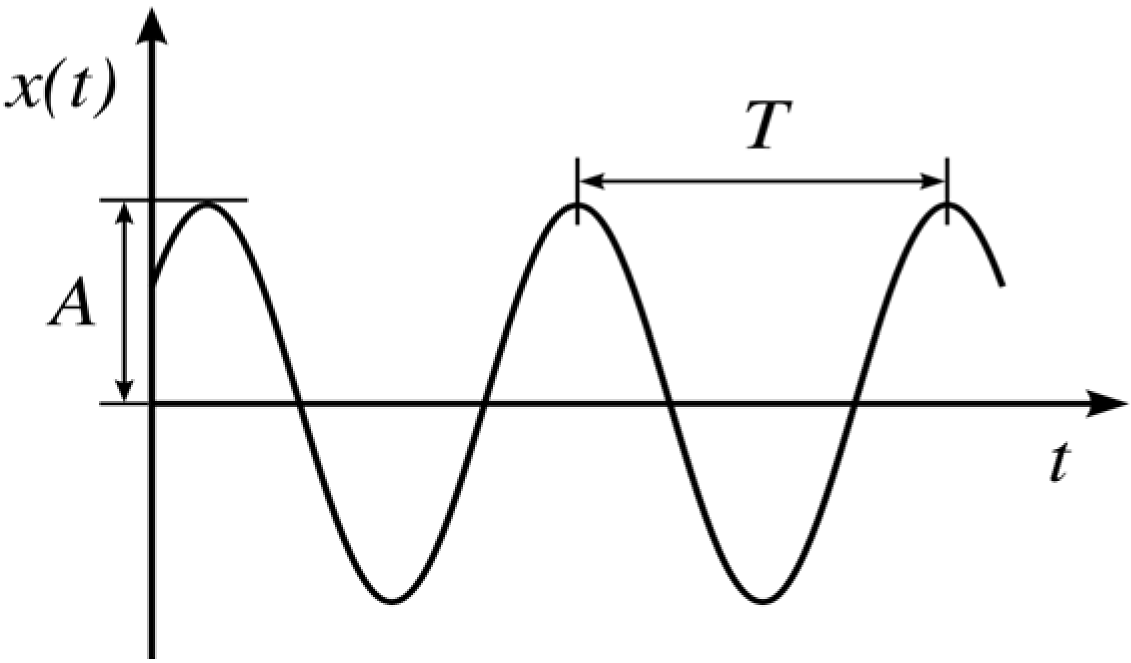

Figure 4. source: Wikimedia

Within the key of C the only recognized chords are built off of the notes of the respective major scale. This causes some chords to be major and some minor, because of the unequal spacing of tones in the scale. To build the standard three-note chord we start with any of the tones of the C major scale, skip one, then use the next one, then skip one again and use the next one after that. The first chord of C major is C-E-G. Because the bfirst interval formed, between C-E is a major third, we call this a major chord, C major. The next chord built in similar fashion is D-F-A. Because the interval between D and F is a minor third this is a minor chord, D minor (Levitin, 2006).

If we graph the sine waves of the three notes, C-E-G, we will notice that all three intersect at a specific point. The intersection point for C will be exactly 2 periods. The intersection point for E will be exactly 2.5 periods. The intersection point for G will be exactly 3 periods. This will hold true for all major chords.

Major chords and minor chords have a very different sound. Even though most non-musicians could not name a chord upon hearing it, or label it as major or minor, if they hear a major chord and a minor chord played one right after the other they would be able to distinguish the two. "And their brains can certainly tell the difference - a number of studies have shown that nonmusicians produce different physiological responses to major versus minor chords, and major versus minor keys" (Levitin, 2006, p. 267).

The sine waves for randomly selected notes will also intersect randomly. It is interesting that only when sounds fit a certain pattern do we find them pleasing, or consonant.

Another, although more complex, application of sine waves is in modeling brain activity through brain waves. One way to record brain activity is with the electroencphalogram (EEG) along with measures of eye movement and skeletal muscle movement. In figure 5 the EEG is highlighted in the dark box.

The heavy dark line highlights rapid eye movement (REM) sleep. There are generally two stages to sleep, REM and non-REM. Most memorable dreaming occurs during REM sleep. The EEG is characterized by rapid, low voltage. This is represented in the graph of the EEG as having a high frequency, or short period, and low amplitude. During this stage of sleep mammals lose muscle tone and are in a near state of paralysis, likely thought to prevent self-injury while sleeping.

The deepest sleep is characterized by a graph with a relatively low frequency and resulting longer period.

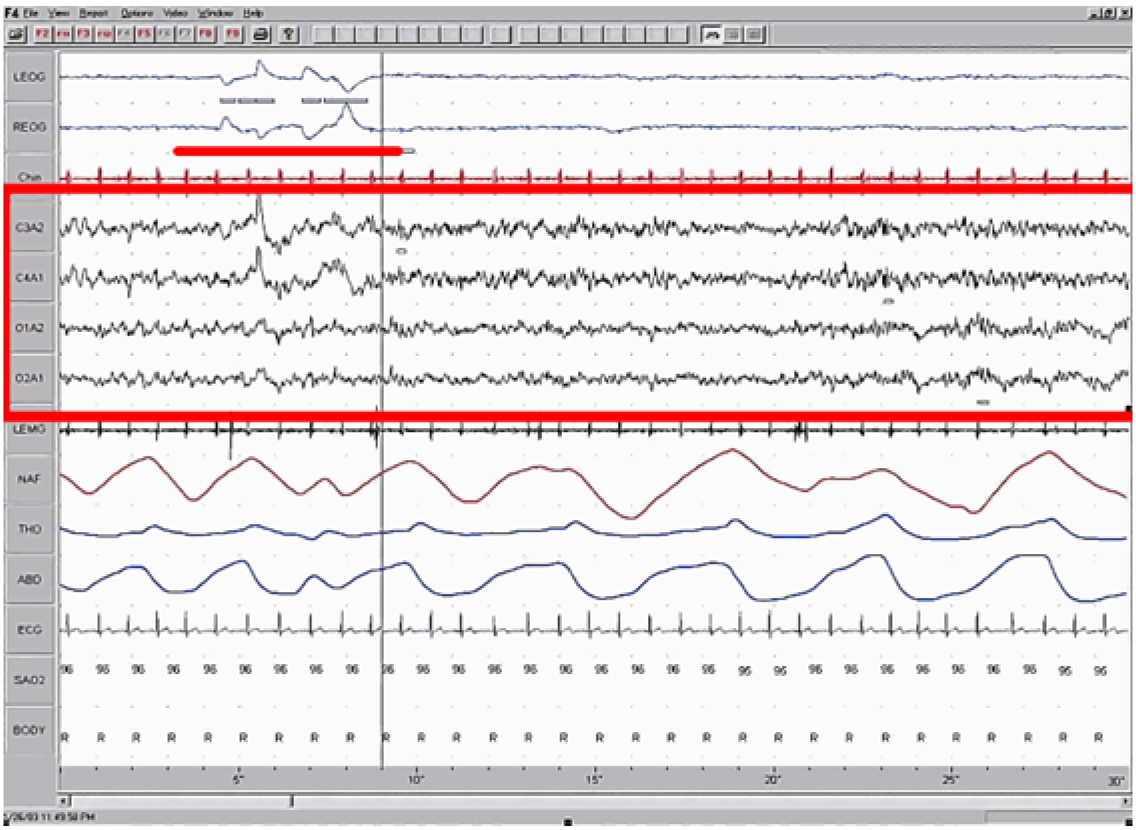

Figure 5. source: Wikimedia

Classroom Activities

Lesson 1

For this lesson idealized data will be used to study the reaction times of various animals, as well as humans. Reaction time is nerve conduction from the foot to the brain and back to the foot. The idealized universal conduction velocity will be 100 meters per second. The students will be given data such as the distance from the extremities to the cortex. For example, from the giraffe's toes to his brain is approximately 5 meters. From a mouse's toe to his brain is approximately 0.1 meters.

Animal	Nerve length	Conduction velocity	Reaction Time
Blue whale	32 meters	100 m/sec.	
Giraffe	5 meters	100 m/sec.	
Human	2 meters	100 m/sec.	
Dog	1 meter	100 m/sec	
Mouse	0.1 meter	100 m/sec.	

Begin the lesson with the question of which will have the quickest reaction time - a giraffe or a mouse? A human or a blue whale? Then have students complete the table remembering that the impulse must travel from the foot or tail to the cortex and back. Once the table is complete have students write and equation. Finally, have students graph their results.

Lesson 2

This lesson will use bags of marbles to represent neurons in the brain. Remind students of the formula for the volume of a sphere: $V = (4/3)*\Pi*r^3$. Have about six different size bags - clear plastic bags work well - in order to show the differing amounts of marbles each bag will hold. Demonstrate the capacity of various bags. Alternatively you may have several students demonstrate this.

Using an average size of 50 micron diameter for each neuron, have students calculate brain volume and required diameter in the following table:

Animal	Neurons	Volume	Diameter
Ant	10,000		
Sea slug	7,000		
Cat	300,000,000		
Dog	160,000,000		
Elephant	200,000,000,000		
Human	100,000,000,000		
Mouse	4,000,000		
Whale	200,000,000,000		
Sponge	0		

Lesson 3

For this activity I use an electronic keyboard and TI-83 or TI-84 graphing calculator. This especially motivates those students with a musical background. Many students have never realized the deep connection between music and mathematics.

First, in order to determine the frequencies of each note in the tempered scale, we start with middle C, which has a frequency of 261.63. Each note up is calculated by the formula $f_n = f_0 * (2^{1/12})^n$ which will produce the following table of frequencies:

C_4	261.63	
$C^{\#}_4/D^b_4$	277.18	
D_4	293.66	
$D^{\#}_4/E^b_4$	311.13	
E_4	329.63	
F_4	349.23	
$F^{\#}_4/G^b_4$	369.99	
G_4	392.00	
$G^{\#}_4/A^b_4$	415.30	
A_4	440.00	
$A^{\#}_4/B^b_4$	466.16	
B_4	493.88	
C_5	523.25	

Have a student, preferably one with a musical background, play several different chords, both consonant and dissonant. Also, have the student play the C major chord.

Using 2 as the amplitude have students insert frequencies for C, E, and G into the y= with each respective frequency in . Set windows on the calculator to y min -2.5, y max 2.5, x min 0, x max Π/265. Make sure students have set the calculators in radian mode.

Have students observe where the three graphs intersect. For C major, the graphs will intersect where C is 2 periods, E is 2.5 periods and G is 3 periods. Have students compare other chords.

Implementing District Standards

The unit addresses several district standards relating to the family of functions which are included in both the Algebra II and Precalculus curriculum. Students will use real and idealized data to represent and model phenomena which will engage them. Specific power standards include:

Identify the characteristics of functions and relations, including domain
and range.
Represent functions and relations on the coordinate plane
Identify an appropriate symbolic representation for a function displayed
graphically or verbally.
Recognize and explain the meaning of x and y intercepts as they relate to
a context, graph, table or equation
Relate the table and graphical representation of a function to its family
and find equations, intercepts, maximum or minimum values, asymptotes and
line of symmetry for that function

Recognize the effect of changes in parameters on the graph of functions or relations

Combine compose and invert functions.

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Web Sites

http://en.wikipedia.org/wiki/List_of_animals_by_number_of_neurons

numbers of neurons

http://faculty.washington.edu/chudler/facts.html brain weights plus lots of other numbers

http://www.bfwpub.com/pdfs/kolbwhishaw5e/ch02part2.pdf

brain weights / body size

http://faculty.washington.edu/chudler/dev.html

brain weights vs age

http://intramural.nimh.nih.gov/chp/articles/lenroot-2006-nbr.pdf

growth of the brain with charts

http://en.wikipedia.org/wiki/Sound_pressure