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Quadratic Regressions and the Catapult Wars

Curriculum Unit 12.04.03
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Introduction

The intended unit of study will explore quadratic functions and is meant to familiarize students with their graphs and the components that define a parabola: the vertex, y-intercept, and x-intercept(s). Students will utilize the regression features of the graphing calculator to determine the equation of a parabola given multiple points along its path. This graphical approach to quadratics is meant to solidify students' conceptual understanding of parabolas. In addition, students will hypothesize how changing features of a catapult affects the trajectory of the projectile and ultimately manipulate the coefficients of a quadratic function in standard form.

Many mathematics texts and resources prompt students to explore the effects of changing coefficients on the shape of a quadratic, but few resources place such a task within a tactile framework. In this unit, students are charged with the task of manipulating a catapult in order to launch a given object a certain distance and height. By analyzing the projectile's trajectory, students are encouraged to hypothesize about and experiment with how to alter the catapult's trajectory. To facilitate student understanding of the larger project, students will participate in mini-lessons designed to introduce keys concepts one at a time. Each of these mini-lessons follows the catapult theme so there is a direct relationship between the conceptual ideas and the manipulatives.

Students are introduced to the unit project by showing a clip from the BBC show Top Gear. The episode focuses on building a catapult-like device that can accurately launch a car through the air and hit a designated target. Students will be asked to describe the shape of the projectile's path. What are the key features on which we should focus? The class will then discuss some of the considerations engineers must make when attempting such a project.

Throughout the unit students are provided multiple examples of catapults to deepen their understanding of the types of questions engineers ask themselves when designing a catapult or to emphasize specific considerations that students should make in their own projects. Each example is meant to inspire students to look at their projects from a different perspective.

This unit is designed for implementation at Wilbur Cross High School (WCHS) and is to be used as an extension to the district's Algebra 1 and Algebra 2 curricula, which extensively study linear and quadratic functions, respectively. The initial application of this unit should occur in Algebra 1 as a way of introducing data collection methods and best-fit models. Initially, students will simply compare how altering arm length, the angle of release, and spring torsion affect the horizontal distance a projectile travels. Students will assess whether the data yields a linear relationship or some other type of relationship. Since the New Haven Algebra 1 curriculum focuses very narrowly on linear functions, students will be asked to assess linear relationships and the factors that indicate that the data are not linear. This will reinforce the concepts of linear functions while also foreshadowing non-linear topics of study in later courses like Algebra 2 and Pre-Calculus.

The second opportunity to implement this unit is in Algebra 2, when students begin to analyze quadratic functions and their graphs. Throughout this unit, students will again experiment with altering the catapult's arm length, the angle of release, and the spring torsion. In this application, however, students will focus on specific points along the trajectory path. First, students will experiment with drawing parabolas given only two points. Through analysis, students will discover that a third point is necessary to deduce the shape of a parabola. Students must hypothesize how to collect these three data points in an accurate manner. Ultimately, students will discover that they can measure the initial launch height, the horizontal distance traveled, and another point that will become known as the canyon point. Through analysis of the data, students will strive to draw conclusions about how altering the three components

WCHS is the largest comprehensive high school in the New Haven Public School district and serves one of the most diverse populations in the city. The school's composition is approximately 89% minority and 72% economically disadvantaged. In 2011, approximately 50% of the 2007 freshman class actually graduated. In an article written by Marian Edelman for change.org about student dropouts, Edelman claims that many of the students that choose to drop out can be spotted as early as fourth to sixth grade and asserts that one of the indicators (in addition to attendance and behavior) is student performance in math class. [1] This unit is designed with the specific aim of engaging students at a new level that will hopefully encourage them to take more interest in their mathematical studies.

Unit Rationale

There has been a recent push for teachers throughout the nation to emphasize science, technology, engineering, and mathematics (STEM) education in high schools. The New Haven Public School district is no different. In an effort to make our district's students more competitive with international students, we must increase interest in and fluency within these areas. Science, technology and engineering have a specific draw for students as each offers a hands-on approach. Mathematics is often approached from a more theoretical standpoint. This unit attempts to combine the best components of each of these disciplines to provide a well-rounded and engaging approach to learning quadratic functions.

The second driving force of this unit is the new Common Core State Standards of Mathematics (CCSSM), in particular the standards for mathematical practice. The CCSS are meant to make students more competitive academically and to help them become college and career ready. The eight standards for mathematical practice push teachers to develop "varieties of expertise" within their students that help them make sense of and communicate about mathematics as a "sensible, useful, and worthwhile" endeavor. [2] The eight standards for mathematical practice are:

1. Make sense of problems and persevere in solving them;
2. Reason abstractly and quantitatively;
3. Construct viable arguments and critique the reasoning of others;
4. Model with mathematics;
5. Use appropriate tools strategically;
6. Attend to precision;
7. Look for and make use of structure;
8. Look for and express regularity in repeated reasoning.

This unit is designed specifically to take students out of the theoretical realm of mathematics so they can experience direct application outside of the typical word problem scenario. Through simple experimentation using the scientific method, students will model with mathematics and persevere to answer the essential question that will be expanded upon in the next section: How do adjustments to various components (arm length, angle of release, and spring torsion) of a catapult affect the trajectory of a projectile?

The Common Core State Standards do not require that each mathematical practice be taught in every lesson, but encourages their regular use, with special emphasis on the first four. This unit of study will meet all eight practice standards.

Unit Objectives

The essential question of this unit is: "How do adjustments to various components of a catapult affect the trajectory of a projectile?" Students will attempt to answer this question by performing a series of experiments. In each experiment, students will change one variable while holding all other variables constant. These variables are arm length, angle of release, and spring torsion. As students experiment and develop models for the trajectory of their projectiles, they should begin to associate specific changes with changes in the coefficients of their functions. Eventually students will be expected to predict how multiple changes to the launch structure will affect the trajectory and associate these alterations to changes in the modeling equation.

This curricular unit corresponds directly with the second and third units of study in the New Haven Public School district algebra 2 curricula. The unit, however, is designed as an introductory approach to quadratic functions and may be used to supplement the last unit of the honors algebra 1 curricula. The second unit focuses on graphing quadratics by calculating the vertex and line of symmetry. The third unit primarily focuses on solving quadratic functions.

Overview and Background Information

Experimentation and Data Collection

The scientific method is commonly used in science classrooms and has served as a guide for how to enter into scientific inquiry. The key steps include:

1. Asking a question;
2. Researching;
3. Constructing a hypothesis;
4. Testing the hypothesis through experimentation;
5. Analyzing data;
6. Drawing conclusions; and
7. Communicating your results.

In this unit, students are provided a question: How do adjustments to various components (arm length, angle of release, and spring torsion) of a catapult affect the trajectory of a projectile? Students will have a chance to examine the catapults, informally assess the various components, and perform basic research before creating a hypothesis about how altering the arm length, angle of release, and spring torsion affect the trajectory of the projectile.

In science, a hypothesis is commonly referred to as an educated guess; that is, a statement that suggests that there is insufficient evidence to declare the statement always true. Hypotheses are established as a guide to the experiment. They provide focus and purpose. Students research possible contributing factors in an effort to predict the outcome of their experiment. Creating a hypothesis also encourages students to think deeply about how to approach the question, ultimately hitting the first standard for mathematical practice: make sense and persevere. Predicting the outcome of an experiment, even if incorrect, helps students make sense of the parameters.

Experimentation is often overlooked as a mathematical practice and immediately skipped for data analysis. Performing an experiment, however, gives a different insight into the interpretation of data as it gives students ownership of the material. An experiment is a scientific procedure used to test the validity of a hypothesis. The key component to an experiment is the procedure. When performing an experiment, it is necessary to have an explicit procedure. The goal of an experiment is not to stumble upon random results, but rather to establish a procedure that can be duplicated to further support the initial results.

Perhaps the simplest way to record data is using a table. In this experiment, students will make use of a t-table; that is, a multiple column table that relates the independent variable (the manipulative) with the dependent variable(s) (the experimental result). T-tables can be recorded horizontally or vertically. Two examples can be seen below:

Angle of Release	0°	15°	30°	45°	60°	75°	90°
Horizontal Distance (yards)	0.5	0.79	1.02	1.24	1.50	1.73	1.98

Arm Length (inches)	Launch Point (0,height)	Ground Point (distance,0)	Canyon Point (horizontal,vertical)
5	(0, 5)	(0.5, 0)	(0.75, -15)
6	(0, 6)	(1, 0)	(1.00, -15)
7	(0, 7)	(1.5, 0)	(1.25, -15)
8	(0, 8)	(2, 0)	(1.50, -15)
9	(0, 9)	(2.5, 0)	(1.75, -15)
10	(0, 10)	(3, 0)	(2.00, -15)

One way to minimize poor data collection is to hold as many variables constant as possible. During any test scenario, it is crucial that only one variable change at a time, so the effects can be directly attributed to those changes. If two variables change, how does the student know which change caused the result? For example: a student launches a projectile and it travels a horizontal distance of five yards. The student then adjusts the arm length and the angle of release and the same projectile travels a horizontal distance of eight yards. Can the three yard increase in horizontal distance be contributed to the change in arm length, the change in the angle of release, or both? How much did each change affect the outcome? Holding all but one variable constant allows the experimenter to limit the factors that may have caused change (remember, there may be variables that change that the experimenter does not realize are being changed). If the student only changes the arm length and the projectile travels six yards, then the student can assume the one yard increase was caused directly by a change in the arm length.

The last item to consider when performing an experiment is to perform multiple trials. Increasing the number of trials reduces the effects of error within the experiment. Just because a catapult launches five yards once, doesn't mean that it will consistently launch five yards. Multiple trials help account for outliers and other data that could skew the results. The law of large numbers states that the average of a large number of trials in an experiment will approximate the expected value. Therefore, the more trials performed, the more likely the experiment will yield results that reflect the true value of the experiment (in this case, the horizontal distance traveled by the projectile). A generally accepted rule of high school science states that three trials are appropriate when performing an experiment. Performing three trials allows you to collect several data points without taking too much time to repeat the experiment.

Analysis of Data, Regression Modeling and the TI-84 Plus

Data analysis will mostly utilize regression modeling on the TI-84 Plus. Once students have collected their data, they will enter it into a table and create a scatter plot. [3]

To enter data into a table:

- Push [STAT], [1].
- Enter the independent variable data into L1 and the dependent variable data into L2, pressing [ENTER] after each entry.
- Press [2 nd], [MODE] to QUIT and return to the home screen.

To create a scatter plot:

- Press [2 nd], [Y=] to access the STAT PLOT editor.
- Press [ENTER] to edit Plot1.
- Press [ENTER] to turn On Plot1.
- Scroll down and highlight the scatter plot graph type (first option in the first row). Press [ENTER] to select the scatter plot type.
- Scroll down and make sure Xlist: is set to L1 and Ylist: is set to L2. To input L1, press [2 nd], [1]. To input L2, press [2 nd], [2].
- Press [GRAPH] to display the scatter plot (you may need to adjust the window). To fit the data to a specific window, press [WINDOW] and enter appropriate values for the extrema and the intervals. For a quick fit, press [ZOOM], [9] to perform a ZoomStat.

To perform a regression:

- Press [STAT], scroll right to display the CALC menu.
- Scroll to one of the regression models and press [ENTER] to select the regression type you want to perform. You will now be on the HOME screen.
- Press [2 nd] [1] [,] [2 nd] [2] to indicate which lists the data is saved in. Press [,] [VARS]. Scroll right to bring up the Y-VARS menu. Press [1] [1] to store your regression in the Y= menu as Y_1. Press [ENTER].

If the above procedure is followed correctly, the calculator will display the results of the regression, including the equation for the selected model that best-fits the data and the coefficient of determination. The coefficient of determination is also known as the r^2 value and indicates how well interpolated and extrapolated data fit the regression model. The coefficient of determination is such that $0 \leq r^2 \leq 1$. Two variables with a high r^2 value $(r^2 > 0.8)$ are said to show a high degree of correlation, meaning you can us e one variable to predict the value of the other. Two variables with a low r^2 *value* $(r^2 < 0.5)$ are said to show a low degree of correlation, meaning one variable is not a good predictor of the value of the second. A moderate r^2 value may or may not indicate that the regression model is a close fit for making predictions. Note that there is not an agreed upon number scale for how well the r^2 value indicates a degree of correlation between variables.

Interpolation is a process used to predict an intermediate value of the dependent variable; that is, a value that lies somewhere between the highest and lowest values of the data. Extrapolation is a process used to predict values outside the given data by extending the regression curve.

Quadratic Functions and Their Key Features

Quadratic functions in standard form are defined as $f(x) = ax^2 + bx + c$, where a , b and c are real numbers and $a \neq 0$. [4] The parent function $f(x) = x^2$ can be seen in Graph A below. The shape of the graph is known as a parabolic curve, or a parabola. Note that parabolas are composed of an increasing interval (as x increases, y increases) and a decreasing interval (as x increases, y decreases). The vertex of a parabola is the point at which the curve transitions from increasing to decreasing or vice versa. The vertex is known as the minimum (Graph A) or as the maximum (Graph B). A parabola is concave up if it opens upward; that is, its end behavior is increasing as in Graph A. A parabola is concave down if it opens downward; that is, its end behavior is decreasing as in Graph B.

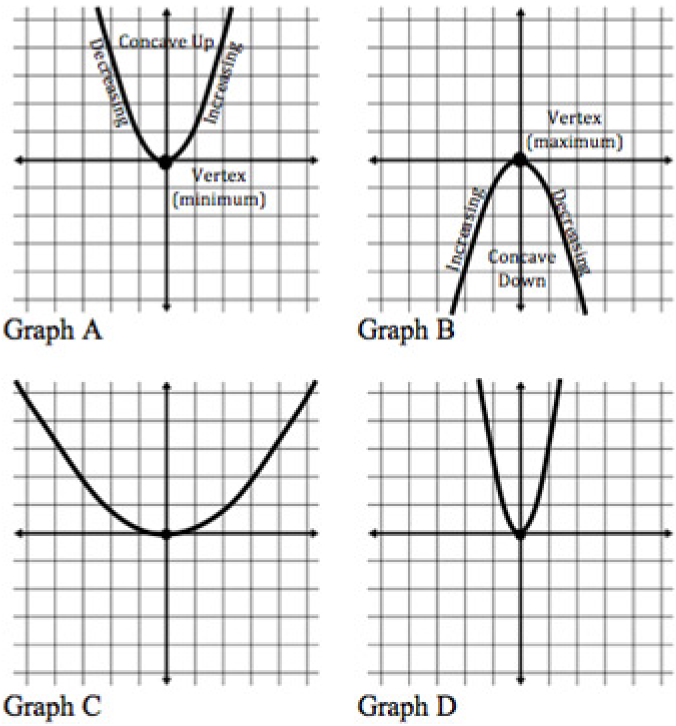


Figure 1
Graphs A and B illustrate key terminology such as intervals in which a quadratic function increases or decreases, the vertex, and its concavity.

Graphs C and D illustrate what happens to the graph of a quadratic function when its leading coefficient, a, is manipulated. When a < 1, as in Graph C, the parabola widens. When a > 1, as in Graph D, the parabola narrows.

The coefficients a , b , and c reveal a lot of information about the shape of a parabola. The leading coefficient, a , indicates the narrowness of the parabola along with the concavity. If $0 < |a| < 1$, then the parabola will appear wider, as in Graph C. If $|a| > 1$, then the parabola will appear narrower, as in Graph D. These changes are known as dilations and occur because the quadratic term, x^2 , is being made smaller or larger, respectively, when it is multiplied by a. A parabola is concave up if $a > 0$ and concave down if $a < 0$.

The constant term, c , is known as the y-intercept; that is, the point where the curve intersects the y- axis. This is easily derived. All points along the y-axis have an x -coordinate of 0 . Thus, the y-intercept can be calculated $f(0) = a(0)^2 + b(0) + c = c$.

The leading coefficient, a , and the linear coefficient, b , can be used to calculate the line of symmetry. All parabolic curves have a line of symmetry given by the formula $x= -b/2a$. The line of symmetry acts like a mirror and marks the horizontal distance at which the parabolic curve switches from increasing to decreasing or vice versa. As such, the line of symmetry can be used to determine the maximum or minimum y-value of the curve. Once the vertical line of symmetry is found, the vertical max or min can be found by evaluating the function $f(-b/2a)$

For example, determine the vertex of the quadratic function $f(x) = 2x^2 + 4x-5$. Therefore, x= - 4/(2(2))= - 4/4= - 1. The minimum value occurs when $x = -1$ $f(-1) = 2(-1)^2 + 4 (-1) - 5 = 2 - 4 - 5 = -7$. Thus, the vertex, or minimum point, of the parabola is $(-1, -7)$.

Finally, all three coefficients can be used to calculate the x-intercept(s) (also known as the solution(s) to the function) using the quadratic formula, which states that the x-coordinate can be calculated as,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Note that the formula includes a ±, suggesting two possible solutions. Because quadratics have one turn at the vertex, the curve can intersect the x-axis in a maximum of two locations, though it may not intersect at all (as is the case when there are complex solutions to the quadratic formula). It is also possible for the quadratic to only have one solution if the vertex lies on the *x-;* axis.

The number and type of solutions to a quadratic function can be determined by examining the discriminant, $b^2 - 4ac$. If the discriminant > 0 , then there will be two real solutions. If the discriminant $= 0$, there is only one real solution. And if the discriminant is < 0 , then there are two complex solutions.

In this unit, students will launch projectiles that have a parabolic trajectory where the vertex does not lie on the x-axis. As such, the equations to the quadratic functions will have negative leading coefficients that result in curves that are concave down (as seen in the figure below). Consequently, this unit will focus solely on quadratic functions with two real solutions. Take note that one of the solutions will be extraneous as the domain is restricted to the first and fourth quadrants of the coordinate plane. Students will study other variations of quadratic functions in later units.

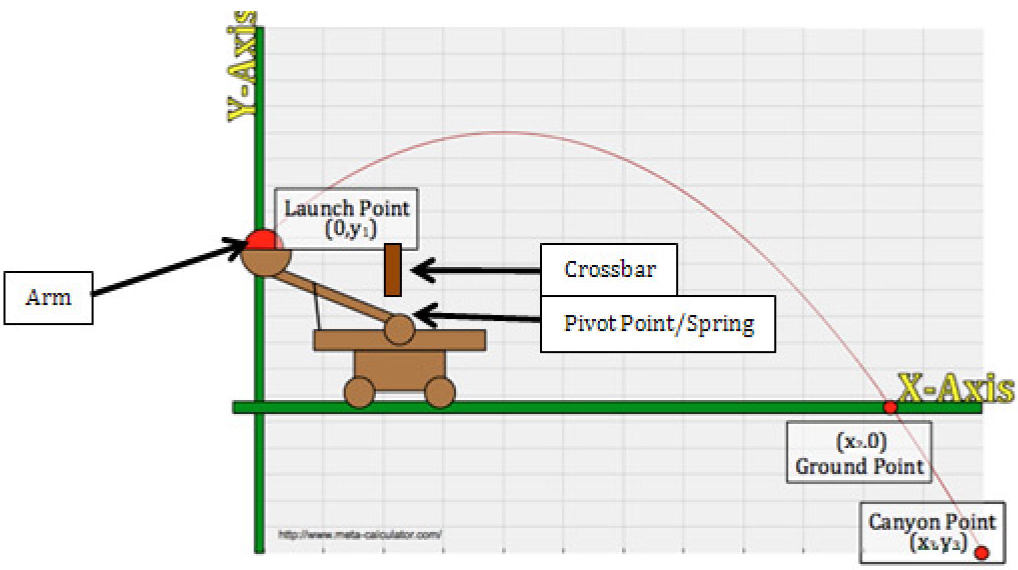


Figure 2
The arm, spring, and crossbar of a catapult. Also labeled are the three measurable test points: the launch point, the ground point, and the canyon point.

Catapults and Trajectory

There are four primary forms of catapults: the trebuchet, the mangonel, onager, and ballista. [5] Students are welcome to explore any of these forms, but this unit is written specifically for the mangonel. The mangonel is comprised of three main components: an arm, a crossbar, and a spring. The arm is a rotating device that rotates about the pivot point. At the end of the arm sits a basket that holds the projectile until launch. Students will perform experiments in which they adjust the arm length, the position of the crossbar (which controls the angle of release), and the torsion of the spring. By individually testing these three variables, students will be able to draw conclusions about how each of these components affects the trajectory.

The arm length will directly affect the speed at which the projectile is launched (the initial speed). The initial speed is calculated as the distance traveled along the arc of the moving arm divided by the time it takes to move from rest to the point of launch. The longer the arm is, the greater the launch arc. An example can be seen below in Figure 3. In this figure, the arm travels a circular path, creating a 90° arc. Therefore, the distance traveled by the smaller black arm is calculated as

$$d = \frac{1}{4}(2\pi r) = \frac{1}{2}\pi r.$$

The longer green arm is twice the length and therefore the distance traveled is calculated as

$$d = \frac{1}{4}(2\pi r) = \frac{1}{2}\pi r.$$

Since the arm covers the larger distance in the same amount of time as it covers the shorter distance, the initial velocity for the longer arm is greater than that of the shorter arm. The arm length also dictates the initial height of the projectile at the launch point.

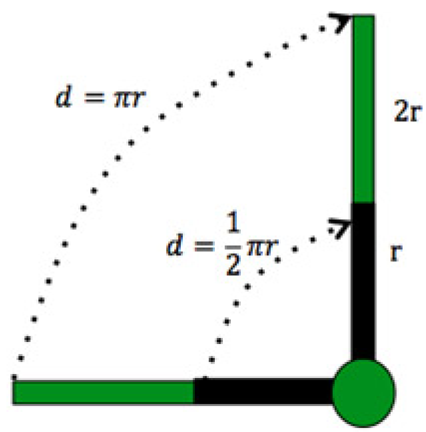


Figure 3
The initial velocity of the longer arm is greater because the arm travels a greater distance in the same amount of time.

Mangonels were typically used as weapons with which to destroy walls (as opposed to launching objects over walls) as they function more efficiently given smaller angles of release. Students will explore how the launch angle affects their ability to accurately predict an experimental equation for the trajectory of the projectile. The angle at which the projectile is released is dependent upon the placement of the crossbar. The projectile is released from the basket once the arm hits the crossbar. Therefore, adjusting the placement of the crossbar allows the user to adjust the angle at which the projectile is launched. Decreasing the angle of release decreases the distance the arm travels. [5]

There are two types of springs: torsion and tension. For this unit, students will explore how changing the torsion of the catapult affects the trajectory of the projectile. Torsion springs are comprised of a rope or band that is twisted around the pivot point (beam). By tightening the spring, students will increase the amount of torsion, which increases the amount of force with which the projectile is launched. Loosening the spring has the opposite effect. Tightening the spring will increase the distance the projectile travels.

Concept List

By the end of this unit, students should be able to explain or define each of the following terms or ideas:

1. Catapult, trajectory and how various components affect the trajectory of a projectile;
2. How to perform a regression to create a best-fit model given experimental data;
3. Domain and range and how context affects the theoretical domain and range differently than the experimental;
4. Extrapolation and interpolation as a method of prediction;
5. Parabolas, their extrema and intercepts;
6. Quadratic Function
7. Slope as an indicator of whether a function is increasing or decreasing;

Teaching Strategies

When having students complete this curricular unit, you may choose a variety of methods. It is recommended that students work in small groups (3 people) so they can collaborate. You may choose to group students of similar ability level or by mixing ability levels. None of the three components makes it significantly more difficult to create a best-fit model, but adjustments to the arm-length are the least complicated to explain when it comes to discussing the affects of the alteration on the equation. If you choose to group by similar ability, give the arm length testing to your lowest level group.

Additionally, you may choose to have groups perform trials for each component of the catapult (arm length, angle of release, spring torsion) or assign specific groups to test individual components. There are, of course, benefits to both strategies.

With the first, every group will collect data, which will provide the class with a much larger set of data. By the law of large numbers, this means the data will more accurately reflect the true value. Note that to use this strategy, all groups must have identical catapults. This will work particularly well if you have chosen to use the suggested catapult kit. On the other hand, it is impossible for each catapult to be exactly identical, so each catapult will result in slightly different results. Experiments are meant to be replicated, though. Using multiple (let's say nearly identical) catapults will allow students to discuss the role error plays in experimentation.

The biggest benefits to the second strategy are time management and meeting the third CCSSM practice standard. By assigning groups to test one specific component, you reduce the amount of time required for experimentation by two-thirds. This also gives students the opportunity to share their data with other groups and to serve as an authority on the subject. This will increase intergroup dependence and require individuals to rely on other students to meet their collective goal. It also facilitates the process of constructing viable arguments and critiquing the reasoning of others (CCSSM Practice Standard 3).

Lesson Plans

Lesson 1: Evaluating and Developing Graphical Representations of Quadratics

Overview

Students will begin their exploration of quadratic functions by evaluating functions and developing graphical representations of these functions. Evaluating the functions to create a table of values, students will graph multiple quadratic functions so they can familiarize themselves with the basic appearance of a parabolic graph. This is a 45-minute lesson.

Objectives

Students will be able to…

1. evaluate a quadratic function using function notation, and
2. graph a quadratic function by plotting a set of ordered pairs.

Warm-Up with Teacher Notes

Students will complete the warm-up handout, which reviews function notation and how to evaluate functions in function notation. The warm-up is meant to help scaffold the rest of the lesson, which will rely heavily on evaluating functions. Students should need no more than 5 minutes to complete this assignment. When the class is done, ask students to quickly go around the room and compare answers. Are there any discrepancies? Allow students to explain the problems to one another. You can assume that some students did not square -1 properly and therefore made a mistake in #3. Have a student review #3 on the board if a lot of the students made this mistake.

Prompt: *f(x), said* "*f of* x," is how we express the value of the function when x is a given value. This notation is known as function notation .$f(x)$ = is equivalent to y = . For example, $f(x) = 3x + 2$, find $f(5)$ In other words, what is the value of y when $x = 5$

f(5) = 3(5) + 2 = 15 + 2 = 17

Let $f(x) = x^2 + 3x - 2.$. Determine the values of…

1. $f(2)$

2. $f(0)$

3. $f(-1)$

Mini-Lesson

Give a mini-lesson in which you review the definition of a quadratic function. How does a quadratic function differ from a linear function? What might a quadratic function look like? Ask the students to develop a hypothesis about what a quadratic function would look like. Students will check their hypotheses in the next activity.

Graphing Quadratic Functions Activity

This is a small group activity. Divide the class into groups of two or three. Each group will be assigned one of three prompts. Ask students to evaluate the function over a defined interval

$$(\text{i.e. } \{-10 \leq x \leq 10 | x \in \mathbb{Z}\})$$

and to create a t-table of values. Students should be familiar at this point with t-tables and how to plot points from a t-table. Step 1 for each group requires the same skills, but offers a slightly different prompt so groups can make comparisons later in the activity.

Each group should be given an oversized piece of chart paper on which to write their function, their t-table, and to draw a graph. Students should follow the prompts on the handout for setting up their graph. Step 2 is the same for all groups.

Once students have completed their graphs, have them display their graphs next to one another on the wall. Students will complete a gallery walk to compare and contrast the overall appearance of the graphs. What do quadratic functions look like? What was similar about each of the graphs? How did they differ? Students will then share their observations with the class during an all-class discussion.

Graphing Quadratic Functions Activity Prompt 1

Hang gliders are small aerial crafts that people usually fly for recreation. You are a photographer at a hang gliding show. You were instructed to take a photo every 10 seconds in order to capture the movement of the hang glider's dive. The dive is modeled by the equation $f(x)=(1/100)x^2 - (25/2)x + 30$, where x represents the time since the dive started and $f(x)$ represents the height of the hang glider.

1. Determine the hang glider's location every 10 seconds for the first 100 seconds of the dive by evaluating $f(0), f(10), f(20), f(30), \ldots, f(90), f(100)$. Record these values in the t-table below.

x	0	10	20	30	40	50	60	70	80	90	100
$f(x)$											

Graphing Quadratic Functions Activity Prompt 2

Catapults were medieval weapons designed to launch an object, usually for destructive purposes. You are designing a catapult that launches a boulder along a trajectory given by the equation $f(x) = -(1/10)x^2 + x + 6$, where x represents time and $f(x)$ represents the height of the projectile.

1. Determine the boulder's position every 2 seconds for the first 14 seconds after it's been shot by evaluating f (0), f (2), f (4), ... , f (12), f (14). Record these values in the t-table below.

x	0	2	4	6	8	10	12	14
$f(x)$								

Graphing Quadratic Functions Activity Prompt 3

You were instructed to design an apparatus for a skate park in a nearby neighborhood. A skateboarder's position on the apparatus is modeled by the equation $f(x) = 2(x-2)^2$, where x represents seconds and f (x) represents the skateboarders height from the ground.

1. Determine the skateboarder's position on the apparatus for the first 10 seconds by evaluating *f (0), f (1), f (2), f (3), ... , f (9), f (10).* Record these values in the t-table below.

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$											

Graphing Quadratic Functions Activity Part 2 (same for all groups)

2. On a large sheet of graph paper, create a graph using the data from the table. Be creative in your representation of the scenario.
3. You will complete a gallery walk to compare and contrast several different quadratic functions. Divide a sheet of paper up into three columns: similarities, differences, and other observations as seen below. Record any observations you have about the graphs of each function. How are the graphs similar? How do they differ? Are there any interesting features?

Similarities	Differences	Other Observations

Homework

You launch a catapult that has a trajectory given by the equation

$f(x) = -(1/2)x^2 + 3x + 5$ where x represents the number of seconds since the projectile was launched and $f(x)$ represents the height of the projectile.

1. Complete the following t-table by evaluating the equation given above.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$										

2. Create a graph that depicts the trajectory of the catapult by plotting the ordered pairs given in the table in question 1.

3. Use the graph to answer the following questions.

a. What is the initial launch height of the projectile?
b. How high can the catapult launch something?
c. How many seconds does it take the projectile to reach the highest point along its trajectory?
d. Along what interval is the projectile's height decreasing?
e. Describe what happens to the projectile during the interval $7 < x < 8$.
f. Is $f(9)$ feasible? Explain why or why not within the context of this problem.

Lesson 2: Analyzing Graphical Representations of Quadratic Functions

Overview

Students will begin learning key terminology and familiarizing themselves with the key components of parabolic curves: the vertex, absolute maximum/minimum, *y-;* intercept, x-intercept(s), and increasing and decreasing intervals along the curve.

Objectives

Students will be able to…

1. identify the vertex and line of symmetry of a parabola, its concavity, the regions in which the function is increasing and decreasing; and
2. determine the domain and range of a parabola.

Warm-Up

1. Graph the function $f(x) = (x + 1)^2 - 3$ on a sheet of graph paper by evaluating the function over the interval $\{-5 \leq x \leq 5 | x \in \mathbb{Z}\}$.
2. Identify the *highest* point along the curve.

3. Identify the *lowest* point along the curve.

Class Notes and Discussion

The prior activity will lead into a discussion on what quadratic functions look like graphically. Class notes should review definitions of parabolas, increasing and decreasing intervals, extrema, domain, range, the line of symmetry, x- and y-intercepts and concavity. Review several examples until students are able to identify each of these components on a graph. Students will practice identifying these components within context in the next activity.

Flashback to Lesson 1

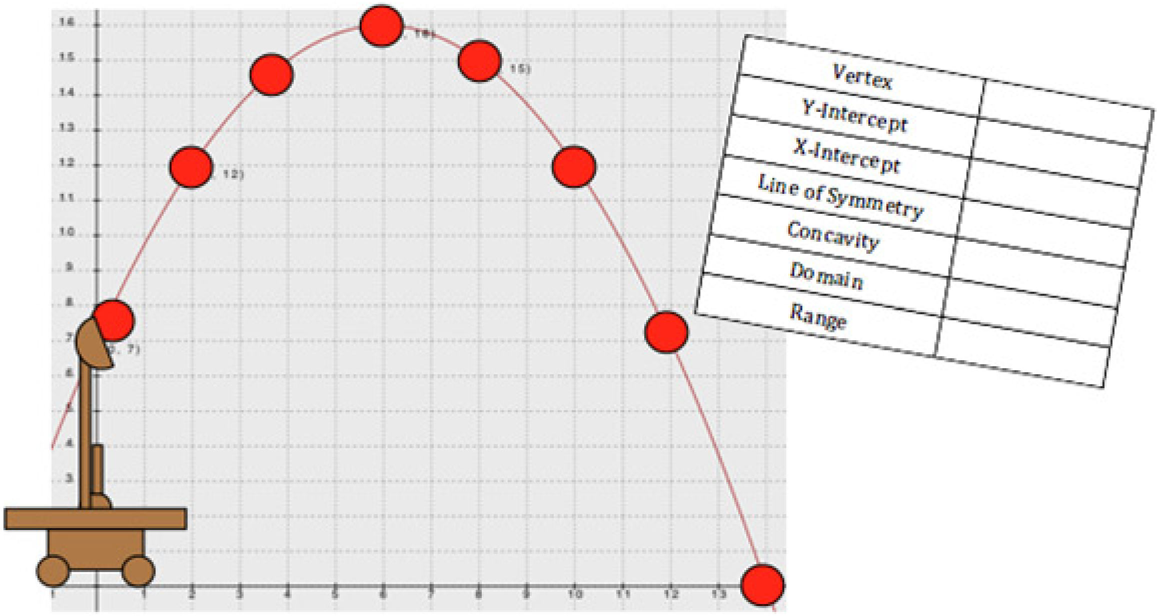
Examine the posters (Hang Glider, Catapult, and Skate Park) each group made during Lesson 1. Analyze each poster, and record the following information in the table below.

	Hang Glider	**Catapult**	**Skate Park**
Interval for which the function is *INCREASING*			
Vertex			
Interval for which the function is *DECREASING*			
Concavity			
Equation of Line of Symmetry			
Y-Intercept			
X-Intercept(s)			
Domain			
Range			

Homework

You launch a catapult that has a trajectory given by the equation $f(x) = (-1/2)x^2 + 2x + 5$ where x represents the number of seconds since the projectile was launched and $f(x)$ represents the height of the projectile.

1. Graph the function on a sheet of graph paper by evaluating it over the interval $\{-5 \leq x \leq 5 | x \in \mathbb{Z}\}$.
2. Identify each component in the graph. Then, record it in the table below.



Lesson 3: Modeling Quadratic Functions

Overview

Students will learn how to perform a quadratic regression in order to determine the equation of a quadratic function given a set of data points. Students will begin interpreting the significance of the vertex, the intercepts, and increasing/decreasing intervals.

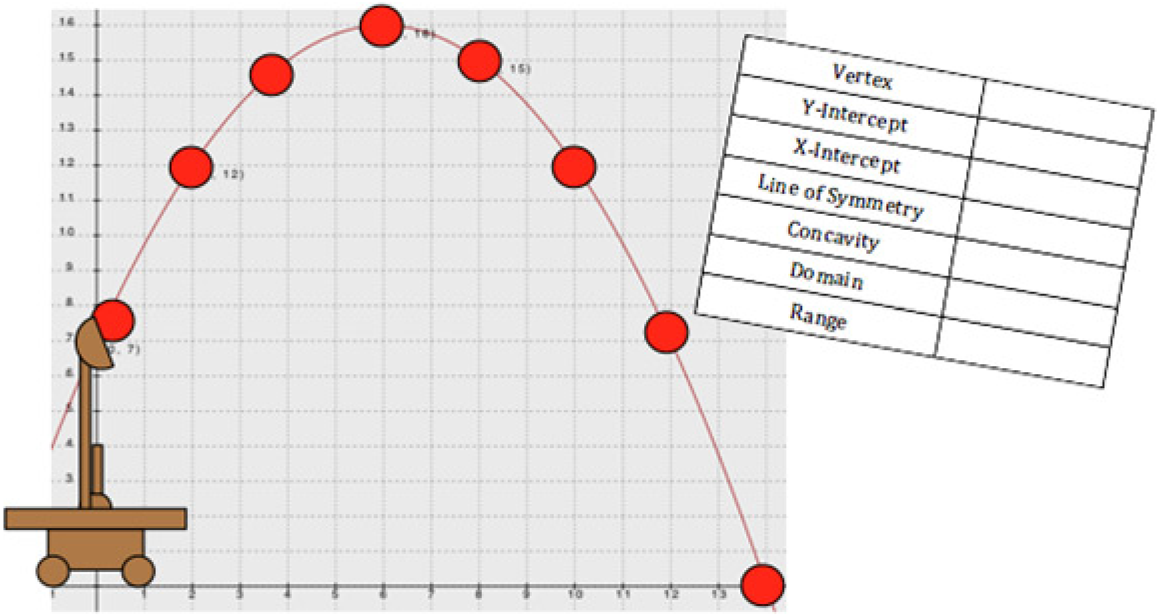
Objectives

Students will be able to…

1. input data into a graphing calculator;
2. perform various regressions to assess what type of model fits a set of data the best and to determine an equation that approximates a model of the data; and
3. use a quadratic regression model in order to interpolate/extrapolate desired data points.

Warm-Up

A catapult was launched and the projectile followed the path seen in the graph. Use the graph to complete the table below. Each axis is scaled by 1.



Graphing Calculator Mini-Lesson

Students will be given a set of notes on how to perform a regression on a set of data. Students will learn about the coefficient of determination and what r^2 values indicate about the fit of the model to the data. Walk students through a couple examples of how to perform a regression by looking at several data sets for practice. Data sets can be found on the Math Tidbits website. It is important to look at non-quadratic models so students gain a richer understanding of how to determine what type of model fits the data best.

Regression Models and Developing a Best-Fitting Curve Activity

Perform linear, quadratic, and exponential regressions for the following sets of data. Determine which model fits the data best by examining the coefficient of determination. Finally, record the equation of each model.

	Set 1	Set 2	Set 3
Data	(-7,1), (-6,13), (-5,25), (-4,30), (-3,42), (-2,45), (-1,50), (0,50), (1,47), (2,48), (3,40), (4,35), (5,25), (6,10), (7,1)	(10,8), (8,6.95), (13,7.58), (9,8.81), (11, 8.33), (14,9.96), (6,7.24), (4,4.26), (12,10.84), (7,4.82), (5,5.68)	(0,179.5), (5,168.7), (8,158.1), (11,149.2), (15,141.7), (18,134.6), (22,125.4), (25,123.5), (30,116.3), (34,113.2), (38,109.1), (42,105.7), (45,102.2), (50,100.5)
Linear r^2 Value			
Quadratic r^2 Value			
Exponential r^2 Value			
Which Model Fits the Data Best? Linear, Quadratic, Exponential			
Equation of Best-Fitting Curve			
Evaluate $f(10)$ using a graphing calculator.			
If $f(x) = 0$, find x using a graphing calculator.			
Are the models strong, moderate, or weak predictors of values?			

Lesson 4: Exploring the Effects of Altering Coefficients

Overview

This lesson serves as the capstone project to this unit. Students will explore how adjusting various components of the catapult affect the equation of a quadratic function. Students will attempt to tie these changes directly to the coefficients of the quadratic model $y = ax^2 + bx + c$. In the end, students will make predictions about what kind of adjustments they can perform to the catapult in order to alter the equation to meet a desired theoretical outcome.

Warm-Up

Show a clip from Top Gear Series 4 Episode 4. This clip introduces students to a car catapult and is a good illustrator of the parabolic trajectory of a catapult launch. Use this time to discuss with students what engineers must consider when designing an apparatus like a catapult. Ask students to consider how parabolas can be manipulated. Transition from this discussion into the Unit Project.

Unit Project

Students should be divided into groups of approximately three to four. Each group will be provided a catapult. Depending upon the amount of time available, the teacher can assign each group to perform each experiment, or can ask each group to complete only one experiment. For a shorter span of time, assign each group to complete only one experiment and then have students share their results. Once all groups have recorded all the data, students will be able to complete the analysis.

Provide a basic demonstration to the class that shows how to use the catapult, how to adjust its various components, and review the location of the three points being collected. Data from the demonstration should be recorded on the board and will serve as the data for the control group. All data collected by groups after altering a component to the catapult will be compared directly to this control data.

The launch point is the point of release. This point is located at *(0,y_1)* , as seen in Figure 1, where y_1 should be the height of the catapult when the arm is vertical. The ground point is *(x_2,0). x_2* is the horizontal distance of where the projectile hits the ground. The third point is labeled the canyon point since students will move the catapult to an elevated surface for the final launch. Explain to the students that whatever surface the catapult sits on serves as the x-axis. Although the floor may initially serve as the x-axis, once the catapult has been repositioned to a table, for example, the table becomes the new x-axis. When the catapult is repositioned onto an elevated surface, the graph's window is changed to include the third and fourth quadrants, or the region in which $y < 0$.

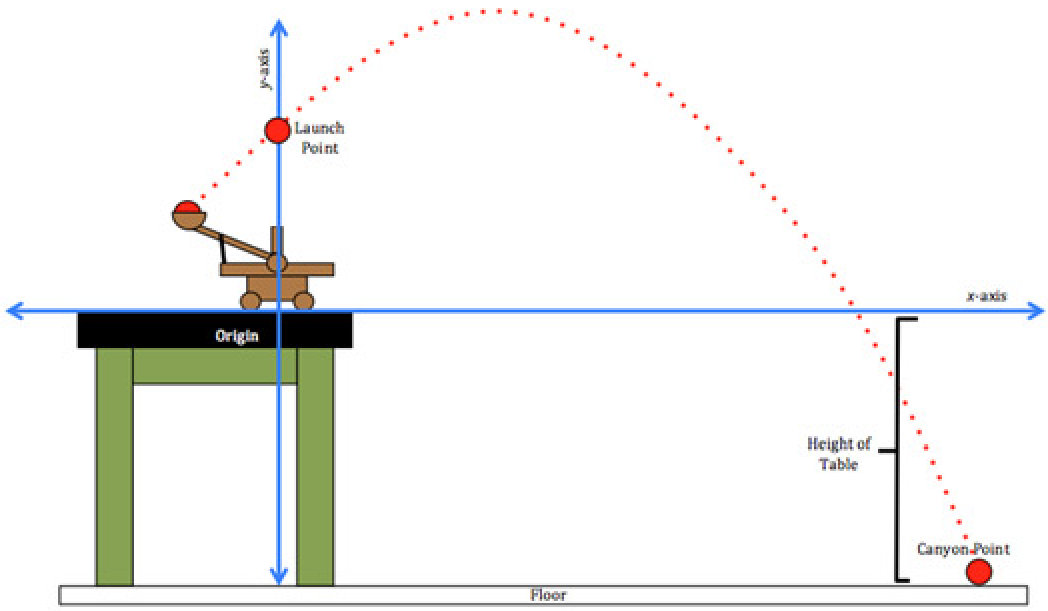


Figure 4
The origin should sit directly below the point of release. Therefore, the origin is tied to the location of the crossbar. The canyon point (x_3,y_3) is located on the floor when the projectile is launched from an elevated surface like a table. x_3 is the horizontal distance traveled from the origin to the floor, while y_3 is the height of the table.

Experimental Procedure

To determine a model for the projectile, it is necessary to collect several data points. For this project, you will perform numerous trials and perform a quadratic regression to determine an equation that approximates the parabolic trajectory of the projectile.

To begin, copy the initial information for all the catapults from those used in the control group in the demonstration. This information should be recorded below. Every group should begin with the same initial information except in ONE (1) instance. The teacher will instruct each group to either vary the arm length, the angle of release, or the number of times the rubber band is twisted. It is essential that students only vary one of these components, otherwise they will not be able to attribute any changes in flight directly to the change in the apparatus.

Have the alterations written down before class begins to make it easier to assign what alterations each group will perform. One group will shorten the arm length, while another will lengthen it. One group should decrease the angle of release, while another increases it. Finally, one group should reduce the number of twists, while another increases the number of twists of the rubber band.

Initial Information (From Demonstration)	Experimental Information (Record Alteration only)
Arm Length: _____ in. Angle of Release: _____° Rubber Band Twists: _____	Arm Length: _____ in. Angle of Release: _____° Rubber Band Twists: _____
Launch Point: (0,______) Height of Elevated Surface: _______ in.	

Experimental Data	Ground Point Horizontal Distance	Canyon Point Horizontal Distance
Trial 1		
Trial 2		
Trial 3		
Trial 4		
Trial 5		
Average		

Follow these steps to measure the vertical and horizontal distance the catapult can launch a projectile. Let x represent the horizontal distance the projectile travels. Let y represent the vertical distance the projectile travels.

1. Set the catapult on the floor. Launch the projectile and measure the horizontal distance to where the projectile hits the floor. Call this point the Ground Point. Perform 5 trials. Record the data in the table above.
2. Set the catapult on a flat, elevated surface (every group should use the same elevated surface for consistency). Measure the vertical distance from the base of the catapult to the floor. Launch the projectile and measure the horizontal distance to where the projectile hits the floor. Call this point the Canyon Point. Perform 5 trials. Record the data in the table above.
3. Determine the average horizontal distance of the Ground and Canyon points.
4. Input the three points (Launch, Ground, Canyon), as seen in Figure 2, into the STATS menu of your graphing calculator. Perform a quadratic regression to determine an equation for the best-fitting curve of the projectile's trajectory.

Have the two groups that altered the arm length compare their three equations (their two plus the control equation from the demonstration). Have the two groups that altered the angle of release compare their two equations with the control equation. Finally, have the two groups that altered the torsion of the rubber band compare their two equations with the control equation. Are there significant changes in the equation, or did only certain coefficients change? It may be necessary to point out to the students that some changes to the coefficients may be insignificant while others are quite large.

Reflection and Conclusion

1. Record the three equations below:

Equation 1	Equation 2	Equation 3

2. Highlight any coefficients that are significantly different. Is there one coefficient that was affected more than the others?
3. Write a conclusion that explains how altering a specific component of the catapult will affect the trajectory equation of the projectile.

Groups should present their results to the class. If the experiments went well, the group that changed the arm length should have equations that are nearly identical with the exception of c, the y-intercept. The angle of release group should notice that a, the leading coefficient, is significantly different. Students should be able to make a connection to the fact that changing the angle of release caused the parabola to widen or narrow, which they know by now is caused by the leading coefficient either being greater than or less than 1, respectively. Finally, the torsion group should have relatively similar *a*- and *c*- values as the angle of release and the initial launch height did not change. The *b* coefficient, however, should differ for each equation.

Refer to the following prompt and your conclusions from earlier in the lesson to answer the questions below. Students will need to consider manipulations of the catapult apart from those that they were assigned in their original groups.

A catapult launches with a trajectory given by the equation $f(x) = -1/4x^2 + 2x + 5$.

1. Identify two manipulations to the equation that would result in the catapult being able to launch further.
2. Identify what adjustments to the catapult would result in the desired manipulations to the equation you mentioned in question 1.
3. If the arm length and angle of release are decreased, explain what will happen to the trajectory of the projectile. What type of changes do you expect to occur to the coefficients of the equation? Explain your reasoning.

The last component of this activity is to have students use what they know about manipulating the catapult to alter their equations in order to hit specified targets. Provide three targets around the room, each of varying distance from the catapults. Ask the students to evaluate the control equation to see if adjustments are necessary. If $y = 0$, then the catapult is positioned correctly. Position one target so no adjustments are needed. Position the other two so students must adjust their catapults to hit the targets. Students should know what type of adjustments will increase or decrease the distance the catapult launches the projectile.

Algebra 1 Alternative

Alternatively to the Algebra 2 approach to the project, students can make alterations to just one component of the catapult and test to see whether altering arm length, for example, yields a linear or non-linear change in horizontal distance thrown. This slight alteration introduces students to modeling without necessarily going to in depth into quadratics, making it a more algebra 1 friendly project.

Notes

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[2] Common Core State Standards for Mathematics, 6.

[4] Ron Larson, "Quadratic Functions," in *Algebra 2* , 246-319.

Bibliography

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[1] Edelman, Marian. "Children Drop Out and Into Lives of Poverty and Imprisonment." Change.org, last modified 1/22/2010, http://news.change.org/stories/children-drop-out-and-into-lives-of-poverty-and-imprisonment. This article discusses many of the issues seen in urban school districts.

[2] National Governors Association Center for Best Practices, Council of Chief State School Officers. *Common Core State Standards-Mathematics* . National Governors Association Center for Best Practices, Council of Chief State School Officers, Washington D.C., 2010. These are the national Common Core State Standards for Mathematics and outline the standards for mathematical practice along with the content standards used in NHPS.

[3] Finding Your Way Around the TI-83+/84+ Graphing Calculator, http://mathbits.com/mathbits/tisection/Openpage.htm (accessed June 3, 2012). This site provides a good introduction to the TI calculators and is a good source of data sets.

[4] Larson, Ron and others. *Algebra 2* . McDougal Littell, 2004. This is the NHPS standard textbook used in algebra 2 classrooms throughout the district.

[5] Gurstelle, William. *The Art of the Catapult* . Chicago Review Press, 2004. This book introduces the history of the catapult, simple directions on how to build a catapult, and the math and physics behind the catapult.

Suggested Student Reading

1. Normani, Franco. "Catapult Physics." Real World Physics Problems, 2012, http://www.real-world-physics-problems.com/catapult-physics.html. This site introduces the different types of catapults and the physics behind them.
2. Gurstelle, William. *The Art of the Catapult* . Chicago Review Press, 2004. This book introduces the history of the catapult, simple directions on how to build a catapult, and the math and physics behind the catapult.

Suggested Materials

1. Classroom set of catapults: http://www.enasco.com/product/SB48339M
2. Small and large graph paper
3. Classroom set of TI-84 Plus graphing calculators

Implementing District Standards

This unit is designed to align with the Common Core State Standards for Mathematics (CCSSM). The CCSSM standards this unit will meet are listed below:

CCSSM Standard	Description of Standard
A-CED.2.	Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
A-CED.3.	Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context.
F-IF.2	Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.
F-IF.4	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*
F-IF.5	Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*
F-IF.7a	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases. a. Graph linear and quadratic functions and show intercepts, maxima, and minima.