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# Using Students' Pictorial Representations to Promote Mathematical Thinking

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## Introduction

I was always "good at math." In elementary school, I was able to memorize basic arithmetic facts. I caught on quickly to the traditional algorithms for addition, subtraction, multiplication, and division. I carried when I needed to carry and borrowed when I needed to borrow by following the set of rules established and communicated to me. As procedures got more complicated, I excelled at following a series of processes as outlined in a textbook and reapplying them to a new set of numbers. After being admitted into an advanced math program, I was able to finish calculus by my sophomore year of high school and test out of the general math requirements for college. It wasn't until I was in graduate school taking a course in "Teaching Mathematics in the Elementary School" that I came to the shocking realization that my problem-solving and mathematical-thinking skills were really quite limited. If asked to solve a math problem without being able to use the tried and true algorithms, I found that I was fairly lost. In my personal math education, I was not taught *why* I did all those procedures I did so well or *how* they worked so well. Unfortunately, I was not alone.

Fortunately, there has been a general trend in math education to focus on more than just the traditional algorithms to help ensure that students understand the "hows" and "whys" of solving a range of mathematical problems. This trend is quite evident with the implementation of the Common Core State Standards (CCSS) in which emphasis during a student's early elementary years is placed on using a variety of strategies to meaningfully represent and understand the concepts of addition and subtraction. [1] If this strong foundation is set, students should eventually be able to use the most efficient and accurate strategies to solve a range of problems. Their mathematical-thinking, problem-solving, and higher-order thinking skills should allow them to interpret and solve real-world problems and be able to express their mathematical thinking. CCSS provides a framework that should ideally ensure that students do not end up like me. Ideally, they will understand *why* they are doing the procedures they are doing and not just plugging in new numbers to the same old algorithm, as I had been accustomed to doing.

While CCSS are well intentioned, teachers at all levels are struggling to get their diverse groups of learners to meet the high expectations outlined by the established standards. Having taught in various urban elementary schools with a variety of curricula, I can say that I have yet to find any one curriculum that adequately helps *students* develop, discuss, and use a variety of strategies to solve problems in a meaningful way, as CCSS demands. Time and time again, I find that, while many components of these resources can be beneficial, there are major voids that do not promote mastering the skills and concepts that students will need down the road. It is no secret that it is extremely challenging to build up students' skills when their foundation has not been established. Being a first-grade teacher of a diverse group of English language learners, I want to be able to do all that I can in order to establish a strong foundation upon which later elementary and high-school educators will be able to build. If the expectations of CCSS can be met, our students will be much less likely to make it all the way to grad school believing that they are exceptional math students unless they are actually doing some critical and mathematical thinking.

## Background

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Currently teaching a class of 26 emerging bilingual first graders in New Haven Public Schools, I am using *The Math In Focus: Singapore Math Curriculum* and have found that it is failing to meet the needs of my diverse population of urban students. One foundational philosophy of the *Math In Focus (MIF) Curriculum* is working with the idea that students transition from concrete to pictorial to abstract representations (see Figure 1) of their mathematical thinking. As with many curricula, *MIF* suggests the use of manipulatives to help facilitate students' concrete representations and, of course, the use of universal numbers and symbols to facilitate abstract representations that can be easily communicated with others at this level of understanding. One component of *MIF* that may not be as common in other curricula is the use of pictorial representations, focusing mainly on number bonds (see Figure 1) in first grade and bar models in upper grades. The number bond consists of two circles being connected with one line each to another circle. This is intended to symbolize the two parts (7 and 3, in Figure 1) used to compose the whole (10, in Figure 1). This symbolic representation of a concept continually proves to be unhelpful and confusing for the majority of my students. The *MIF* curriculum makes the claim that "clear and engaging visuals that present concepts and model solutions allow all students regardless of language skills to focus on the math lesson." [2] From my experience, these pictorial representations may help some students, but have been a source of confusion for the majority of my emerging bilingual students, regardless of whether or not the instruction is provided in their native language.

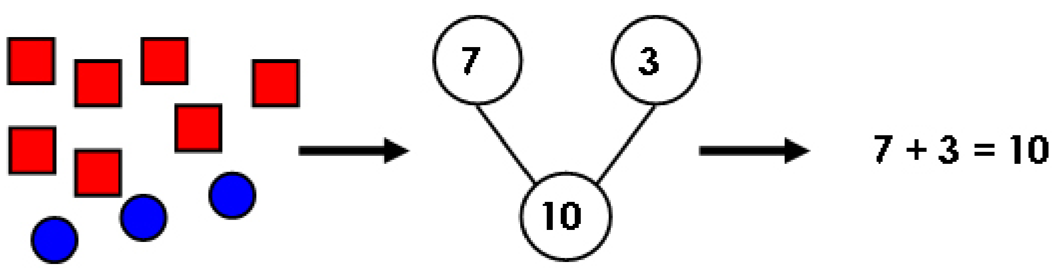


Figure 1: Concrete (blocks) --> Pictorial (example of a number bond) --> Abstract

Throughout the *MIF* curriculum, the order in which different concepts are introduced, the manner in which they are presented, and the tasks given to the students make it difficult for them to master math CCSS. For example, an initial focus is placed on teaching students how to use curriculum-prescribed number bonds rather than developing strategies to add or subtract. The later focus on using strategies to add and subtract is limited and teacher-centered. The formatting of the curriculum suggests that the teacher should explicitly teach and "model" the strategies; and then, through guided and independent practice, the students should use the prescribed strategy. The addition strategies include "counting on" and "using number bonds." Meanwhile, the subtraction strategies include "taking away," "counting on," "counting back," and "using number bonds." No other strategies and no suggestions to have students share their ideas are presented in the curriculum. Furthermore, the tasks that students are asked to do generally involve simply recording the answer. The suggestions in *MIF* have done little to lead to *students* to effectively develop, discuss, and choose efficient strategies as needed to meet the math CCSS.

Despite priding itself on mirroring the practices used in the mathematical powerhouse of Singapore, the *MIF* curriculum is like the majority of the math curricula sold in the United States, as it uses a behaviorist approach. I believe that this heavy reliance on behaviorist theory is one component that is doing a huge disservice to our students. The presumption of behaviorist theory is that our students are "empty vessels" and that our job, as teachers, is to fill them up with facts. While I have certainly met my share of teachers who will openly state that "our students come to school with *nothing* ," I also know that the majority of the teachers I've worked with believe that their students' background knowledge, experiences, and interactions with others are all powerful learning tools. If this is the case, why do we continue to teach math in a way that assumes that the only way for the child to learn is through the direct teaching of skills by the teacher? Those of us who believe children construct knowledge through meaningful experiences and through interactions with peers and adults have a more constructivist or social constructivist view of learning. Most math curricula do not take these views into account, so there is room to expand curricula and classroom practices in order to foster a constructivist approach.

While agreeing with and working with the understanding that students benefit from understanding concrete representations before pictorial representations, and pictorial representations before abstract representations, I would like to argue that using student-created pictorial representations will help to further promote children's mathematical thinking. These representations would certainly be more meaningful to the students given that they will be the ones involved in developing them. Being given the task to focus on representing the process or strategy used instead of focusing simply on recording the product will help students to use metacognitive skills and further construct their concepts of addition and subtraction. Also, pictorially representing the *process* may help to facilitate the transition to the abstract representations of the symbols ("+" and "-") that represent processes. Furthermore, having a pictorial representation of their process will likely help students with limited language skills or those struggling to share and explain their strategies to do so. Through sharing their strategies with their class, all students will be exposed to a variety of strategies that they may be interested in trying to implement, while also having the opportunity to compare strategies for their accuracy and efficiency. Likewise, teachers will be both enlightened as to where their students stand with their problem-solving skills and provided with more information that they can subsequently use in order to challenge students or to help uncover misconceptions that may have otherwise gone unnoticed.

This unit also encourages teachers to hold more asset-based perspectives, rather than deficit-focused perspectives: it attempts to inspire teachers to help each student to use and represent what they already know as a way to help improve their mathematical thinking. A review of research on and examples of student-created pictorial representations will help teachers to look at students' "drawings" through a new lens. Ideally this unit will also help to spark professional conversations about how what we learn from students can influence our instruction and differentiation. Through providing ideas on how to facilitate using students' pictorial representations during math—as well as strategies on how to help students transition from concrete, to pictorial, to abstract representations of addition and subtraction—I hope to provide tools that will result in students being better equipped to meet the high Common Core State Standards. Ultimately, I also hope to influence changes in practice that will result in students being able to independently use higher-order mathematical thinking when approaching math problems throughout their school years and beyond.

## Objectives

The collection of research, teaching strategies, and sample activities that follow are not intended to be merely part of one collective unit. Rather, teachers can use them to promote mathematical thinking throughout the year. By implementing many of these strategies to supplement current curricula, teachers will be better equipped to meet the demands of the math CCSS, particularly within the strand of "Operations and Algebraic Thinking," as well as Mathematical Practices." [3]

Rationale: Why use students' pictorial representations?

When considering the need for students to think mathematically, most experts agree that language is an absolute necessity. As NYU professor of journalism and mass communications Mitchell Stephens notes, "to be human...is to use and understand words. Most of what we can sense, feel or imagine we can express through language." [4] However, the question arises: Do we need language to be able to think? Without language, many would argue that thinking, particularly higher-order thinking, is simply not possible. According to American cognitive psychologist Elizabeth Spelke, humans' unique communication tool of language not only allows us to exchange information with other people, but also serves "as a mechanism of communication between different systems within a single mind." [5] In accordance with this theory, if we want to ensure that our students are thinking mathematically, we inevitably need to make sure that we are helping facilitate the language that allows these types of thought processes to occur.

Yet, if language is so important, why not just ensure that students are given ample opportunities to develop their spoken or oral language? Educator Dr. Susan Sheridan has done integrated research dealing with markings from 1.9 million years ago and "neuroconstructivist" theories. Connecting these theories to the present, she has suggested:

> Since humans still use images to express thought, and, in fact, require images to understand their most complicated, abstract thoughts, it is clear that words have not replaced the power of drawings to express meaning, and, most probably, never will. [6]

Based on the above observations and conclusions, we can certainly work with the assumption that written and spoken language can work in tandem to help further develop children's language skills, as well as their mathematical thinking skills.

Having established that productive language (including speaking and writing) can certainly be capable of helping to promote mathematical thinking skills, it is important to look at how this concept is reflected and supported within our standards and curriculum. Within the domain of "Operations and Algebraic Thinking" of the CCSS, the first-grade standards involve students being able to "solve word problems...by using objects, drawings and equations...to represent the problem." [7] The philosophy of the *Math in Focus* curriculum closely mirrors this standard by approaching solving problems concretely (with objects), pictorially (with drawings), and abstractly (with equations). [8] While I applaud *MIF* curriculum's attempt to scaffold instruction by implementing components of the Concrete-Representational-Abstract (CRA) Instructional Approach [9] , its use of number bonds limits or impedes students' learning and conceptual understanding. When given the outline of the number bond and being asked to fill it in, students are simply not representing their own concrete understanding, and using this "pictorial representation" does nothing more than fail to support children's mathematical thinking. As a result, the number bonds become just another circle to fill in on a worksheet, which founders of the international Children's Mathematics Network, Worthington and Carruthers, criticize as being "to mathematics that 'painting by numbers' is to art." [10] They prevent children from representing their own mathematical thinking and, therefore, limit the power that pictorial representations can have in helping a student understand the transition from concrete to abstract representation.

When considering the problem that the majority of math curricula do not have adequate opportunities for students to effectively transition from concrete to abstract representations in a way that preserves their personal, mathematical thinking, I would like to suggest taking the lead from the practices of emergent literacy. As is done with emergent literacy, in emergent mathematics emphasis should be placed on students creating representations that are meaningful to them. In three separate studies looking at students' representations in problem solving, Woleck, Nicol and Saundry, and Smith found that first-, second-, and third-grade students, respectively, used their pictorial representations as functional tools for problem solving or as a way to represent their thinking or a solution. [11] When encouraging students to use their own pictorial representations, whether used as placeholders for concrete objects or as a form of communicating, teachers will inevitably help to facilitate mathematical thinking and understanding. While using students' pictorial representations at any level can be beneficial, evidence suggests that it is the most critical in the early elementary years.

For those who have limited experiences with emergent writing or with having students develop their own pictorial representations, it may be challenging to know what to expect from early elementary students' self-created representations. According to Carruthers and Worthington, within mathematics students may be creating and reading back dynamic, pictographic, iconic, written or symbolic marks. Dynamic representations are marks that students make that mainly show a change or activity. When students provide representations that are drawings resembling something that is in front of them, they are making pictographic marks. Iconic marks differ in that they use one type of iconic form (often tallies, circles, stars, etc.) to represent each item when counting. When students use written graphics, they are writing out their thoughts (or approximations of such) using letters and words. Finally, students can also use symbolic marks, which involve the abstract numbers and symbols of operations earlier referred to. While most children's graphics will develop in this general order, these areas are not necessarily sequential or mutually exclusive. [12] While to the inexperienced / untrained eye, some children's marks (see Figure 2) may look quite idiosyncratic, it is important to keep in mind that if you listen to the child's explanation of his or her representation or observe his or her process in creating one, you will likely uncover a wealth of thinking.

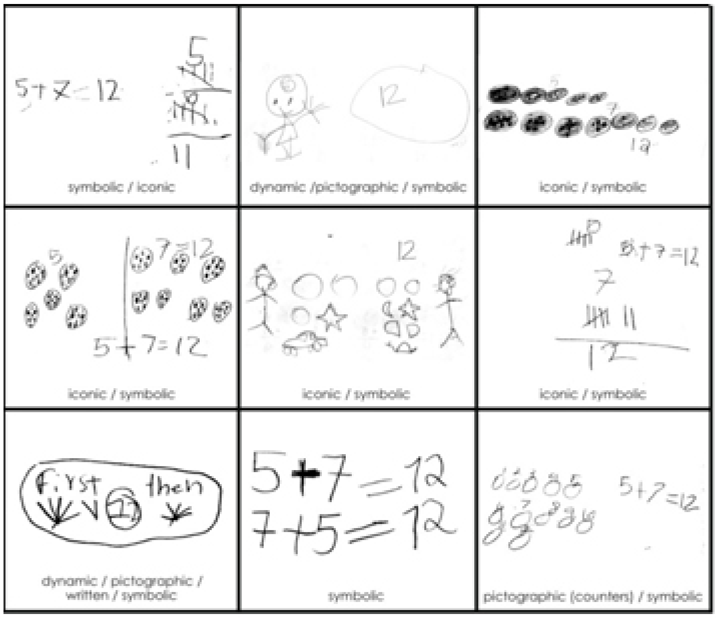


Figure 2: Samples of students' pictorial representations for the story problem:

María has 5 cookies. Alondra has 7 cookies. How many cookies do they have in all?

Of the variety of marks outlined above, the form or combination of forms that a child uses are influenced by his or her past experiences and background knowledge. This prior information may be from home or school or from interactions with peers. By bridging this information, the processes of understanding and finding meaning in the gradual and newly acquired information on abstract symbols will help develop a child into what Carruthers and Worthington have coined as "binumerate": the child understands how to communicate his or her mathematical thinking in two different ways. [13] This idea of using what the students bring to the classroom (regardless of how closely it aligns to our final goal) abandons the common deficit-based perspective, and focuses instead on an asset-based approach, which will also be a cornerstone to facilitating mathematical thinking skills.

## Teaching Strategies

Early elementary teachers can do various things to facilitate opportunities for their students to use their own pictorial representations, make more meaning of their written mathematical systems, and improve their mathematical thinking. To develop a culture in which children represent their mathematical thinking, certain components must be present in terms of the physical environment, the classroom community, the styles of interactions, instructional practices, routines, and professional communities. Simply telling students to "draw what they are thinking" is not enough.

**Math-rich Environment**

First of all, the physical environment of the classroom should be rich in mathematical content. In a rich mathematical environment, a variety of manipulatives, whether commercial or found, are provided as concrete learning tools. Throughout the classroom, one can see number lines, calendars, graphs and charts on display at a level that is accessible to children. As Carruthers and Worthington note, providing a "graphics area" that includes not only literacy-based, but also math-related resources is critical to the math-rich environment. This area should include writing and drawing tools, paper of varying sizes and shapes, clipboards, graph paper, forms, lists of children's names, rulers, calculators, measuring tape, check books, shapes, clocks, etc. [14] In addition, in (most likely preschool and kindergarten) classrooms that have role-play areas, strategic placement of notepads by telephones, clipboards by doctor/police/ firefighter areas, calendars on the refrigerator, and shopping lists and coupons in the store are just a few more ways to create a math-rich environment. Of course, simply providing math-related objects and realia does very little to develop students' mathematical thinking if a nurturing community that embraces children's mathematical thinking and representations is not established and maintained.

**Mathematical Community**

To begin, if the physical resources noted above are in a teacher's classroom, he or she must pay attention to make sure students can use them to help build their understanding of mathematical concepts. For example, the math manipulatives should be readily accessible to students, and limitations should not be put on which manipulatives students can use for which math problems. Rather, a teacher can facilitate discussions as to how students use different resources and how effective and efficient each of them are. The displays and data placed on the walls should be meaningful to students and caution should be taken to ensure that students' representations carry as much weight as conventional representations. The "graphics area" described above should be made available not only during math time, but also during any sort of "free time." As "free time" dwindles across school districts, we have to get creative as teachers and create "mobile graphic areas" and provide access to these resources at any spare moment that we can find: at arrival and dismissal, when students get done early with their breakfast or lunch, and at recess or during "Fun Friday." Finally, and perhaps most importantly, when these moments are provided, be sure to pay attention to what students develop during this time.

*Interacting with Students*

When we listen and carefully observe what students are thinking, we not only show that we value children's thinking and graphical marks, but we also get significant insight into what students' mathematical understandings are and how they can be valued and further developed within our classrooms. While allowing students to use their own pictorial representations is very meaningful for them, if we are not listening carefully to what students are saying and what their drawings are representing, we can easily make inaccurate assumptions as to what a child knows or doesn't know. A Primary Teacher Associate with NRICH (a team of teachers supported by the University of Cambridge aiming to enrich the mathematical experiences of all learners), Bernard Bagnall, pleads for caution on making assumptions when looking at children's representations of mathematics:

> We need to consider the child's ability to communicate (and the opportunities we provide for communication), the child's understanding of his or her own mathematics and to bear in mind that this may not necessarily, at this stage, be totally in line with 'school maths.' [15]

By paying close attention to students' "self-talk" and talk among peers as they developed their mathematical representations, Woleck noted how the language attached to these rich pictures provides significant insight into the students' mathematical thinking and understanding. Through these informal and spontaneous conversations, "children came to question, debate, defend, clarify, and refine their mathematical understandings." [16] Of course, in order to nurture these types of interactions, a teacher must be willing to allow students to talk to one another and to think through their process out loud. Also, circulating and listening in on these conversations, focusing on students' mathematical thinking, will play a valuable role in assessing where students are and providing additional opportunities for discussion.

In addition to providing opportunities for these student-initiated discussions, teachers should carefully plan and adapt lessons to facilitate discussion that takes advantage of the mathematical knowledge and experiences children bring to the classroom. A "Capacity Building Series" publication distributed by the Ontario Ministry of Education gives several suggestions about how to facilitate "math talk" that will encourage and foster children's growth in the understanding of mathematical concepts in the early elementary classroom. One very important component is allowing students to talk about their mathematical thinking after they have worked through solving a problem. During this time, the teacher encourages students to share out their variety of strategies, justify their solutions, and make connections between them, while working to facilitate the development of generalizations of mathematical concepts. [17] This same article proposes the use of Suzanne Chapin's "Five Productive Talk Moves" to further facilitate meaningful discussions around mathematical thinking:

> 1) Revoicing—Repeating what students have said and then asking for clarification ( *So you're saying it's an odd number?* )
> 2) Repeating—Asking students to restate someone else's reasoning ( *Can you repeat what he just said in your own words?* )
> 3) Reasoning—Asking students to apply their own reasoning to someone else's reasoning ( *Do you agree or disagree and why?* )
> 4) Adding on—Prompting students for further participation ( *Would someone like to add something more to that?* )
> 5) Waiting—Using wait time ( *Take your time...We'll wait...* ) [18]

Regardless of the curriculum a district is using, facilitating discussions using some of the above guidelines can be extremely valuable in helping students share their mathematical thinking and more deeply understand mathematical concepts by seeing and hearing about how their peers have solved similar problems. Of course, if and when students have a challenging time articulating their mathematical thinking, using their pictorial representation can help them communicate their ideas and provide others with visual aids to understand their process.

*Bansho (Board Writing)*

When students are sharing their mathematical thinking, the classroom becomes an environment in which student learning is consolidated and both individual and collective mathematical thinking is occurring. One instructional strategy that is used in Japan and, more recently, in Ontario, is referred to as "bansho" or board writing. The Ontario Ministry of Education has adapted the Japanese bansho and made suggestions for implementation to further facilitate children sharing their mathematical ideas in their classroom community. To plan a "bansho" lesson, teachers should use what they know about their students and the grade-level standards to develop an objective, choose a problem that is aligned with this objective, and anticipate each part of the lesson. The problem-solving lesson itself consists of the three parts: before, during, and after students solve the problem presented. Before working on solving the problem, the teacher activates students' mathematical knowledge and presents the problem to the students, as well as records it on the board or chart paper. Students identify information provided and needed to solve the problem, and this information is recorded on the board. Next, students are provided with paper on which they can represent how they solve the problem and record their solution. During this time the teacher observes and records students' solutions, as well as facilitates discussions among students in anticipation of the final parts of the lesson. After students have solved the problem, the teacher guides a class analysis and discussion of solutions, during which time students' work can be sorted and reorganized on the "bansho" in order to draw attention to commonalities and progression towards the objective. During this discussion, the teacher should annotate on or around the students' work in order to explicitly expose students to conventional, abstract representations. Following this discussion, the class collectively develops and records a summary or highlights the key ideas, strategies and representations as they relate to the objective. Finally, the teacher provides students with some additional practice problems to which they can apply their developing mathematical understandings. [19] While this is not a format that would necessarily be used for every math lesson, I believe the bansho instructional strategy is an ideal framework for solving story problems. It allows students to represent their own strategies and solutions, promotes a mathematical community that values children's thinking, facilitates rich discussion and conversation around mathematical thinking, and provides opportunities for the teacher to effectively and appropriately model alternative ways of representing mathematics.

*Modeling vs. Examples*

With the above "bansho" instructional strategy, the teacher does not provide examples of how to record a solution using an equation before students work on solving the problem. Rather, the teacher waits until *after* students have solved the problem to annotate additional models of representation that can be used. This format and the timing of introducing abstract symbols to students are very much in line with the philosophy and suggestions presented by Carruthers and Worthington. They highlight the confusion that often arises between the terms *modeling* and *examples* and caution that modeling should not be interpreted as providing direct examples:

We knew that we needed gradually to introduce children to standard symbols and

various labels but when we provided an example at the beginning of a lesson –

intending to offer one possible way – the children copied exactly what we had done

with limited understanding. [20]

When providing direct models that may be increasingly abstract for children, teachers should display models along-side with and in reference to children's own representations in order to demonstrate that they value what students are bringing to the classroom as well as connect this new type of representation to something that is meaningful to the students. As students get older, it will become more and more necessary to provide representations that can communicate with a broader audience, but particularly when approaching new (and potentially more complex) problems, students need to understand their own process and style of representation first.

*Cognitively Guided Instruction and Solving Problems Using "Just-Right" Numbers*

As students develop their mathematical understandings at different rates, differentiation is needed in order to ensure that students are able to approach solving the problems provided, but that they are also adequately challenged so that new learning can take place. One such differentiation method often used by teachers is to provide different students with completely different story problems. While this method can be rather time-consuming, challenging to manage, and ineffective in helping students deal with the variety of problems as outlined by CCSS (such as adding to, taking from, putting together, taking apart, and comparing), [21] there is another potential way to differentiate story problems. As part of the Cognitively Guided Instruction approach [22] (which would likely support many of the strategies outlined above), this differentiation strategy is known as "Using 'Just-Right' Numbers." Once established in the classroom setting, it can be a very appropriate instructional strategy for students with a variety of mathematical understandings, and it is feasible to facilitate. With this strategy, teachers determine what type of story problem they would like to have students work with and write a story problem frame. A basic example of a "joining" [23] frame would be: "Raúl had___ apples. Jonathan gave him ___ more apples. How many apples does Raúl have in all?" Then the teacher develops sets of numbers that will meet the needs of his or her variety of students. For example, one student may need to work on the concept of simply putting together two quantities, so the set of 2 and 3 would be manageable for him or her. Meanwhile, a group of students may be working on combining sets that are greater than 10 and determining how to recycle their fingers, so a set of 8 and 5 might be appropriate for them, or 5 and 8 if an emphasis is needed on adding more efficiently by counting on starting with the greater number. More advanced students may be working with understanding place value, in which case a set of 18 and 30 might be appropriate for them. By using the same story line, this story problem can be introduced to the whole class, using the basic set [2, 3] as a placeholder. This way everyone can hear an example of the story problem, visualize what is going on, and identify what is known and unknown before moving forward with developing a strategy to solve the problem. Depending on the needs of the students, all students can start with the basic set and work their way through progressively more challenging sets of numbers, the teacher can help guide students to the set that is most appropriate for them to start with, or the students can select which set of numbers is "just-right" for them. Most importantly, with the implementation of this differentiation strategy, the teacher engages all students in the problem-solving process, and they are able to use strategies and represent them in ways that are most meaningful to them.

*Establish a Routine for Solving Story Problems*

As with all frequently used procedures, it is very beneficial to develop a routine for approaching and solving story problems. The anchor chart shown in Figure 3 illustrates the steps that I have taught my first-grade students to follow when solving story problems, and they can be adapted to accommodate other grade-levels and classes. The research that I have done in preparation of writing this unit has both confirmed the method that I have used, and revealed new ideas as to how it can be revised and improved upon. It is best to spend at least one lesson on each "step" of the problem-solving process towards the beginning of the year and re-visit individual steps as needed. While focus may be on one step or another based on the lesson objective or needs of students, be sure to go through all steps to reinforce what students should do when they are solving problems independently.


Figure 3: Anchor chart used to help establish a problem-solving routine

At the beginning of the year, as many first-grade students are still emergent readers, the first step is to listen to the story problem. The focus during this step is to visualize what they hear or read (which is also practiced during literacy) in order to make more sense of the story problem. Next, we work on retelling the story problem using our own words. I encourage the students to keep track of what they already know from the story and identify what it is that they need to find. Since I have been using *MIF* , I have attempted to use the "number bond" as more of a graphic organizer to help students organize their information. When beginning a story problem with first graders, we often do the first two or three steps as a whole group. As students become more independent, I release them to do the third step and even the first and second independently. However, steps four through six are always done independently or with a partner. A significant number of lessons are spent identifying different strategies that students are using, as students continue to develop more strategies and more efficient strategies throughout the year. While the fifth step on my anchor chart is listed as "record," based on my expectation that students represent the thinking they used to solve the problem, it should be labeled as "represent." If students are using "drawing" as a strategy, the fourth and fifth steps may be combined. Because students already have been exposed to a significant number of worksheets and been requested to record things that are not necessarily meaningful to them, representing their own thinking has proven to be challenging for them. By very mindfully observing and listening to students (as discussed above), I have been able to model different ways that individual students can communicate their thinking and strategies. I usually do this both with spoken words and through pictorial representations, and I would encourage other teachers to do the same as a means of helping students become comfortable with representing their thinking independently. One common question that comes up is from students who "just know" the solution to a problem. I encourage them to try to represent that on their paper, as well as represent how they can *prove* that they are correct. This leads us to the final step, which is checking their work. As their peers share more strategies, students often will check their work by using a different strategy to see if they arrive to the same solution. While this routine may feel lengthy, it helps to ensure that students are, indeed, solving problems in ways that are meaningful to them and, thus, improving their mathematical thinking.

*Professional Learning Communities*

Finally, one critical part of maintaining a mathematical community is ensuring that all educators (paraprofessionals, teachers, parents, administrators) have the opportunity to discuss and develop their own understandings of mathematical processes, as well as the mathematical thinking that our students are representing. This sort of engagement can take place in a variety of professional learning communities, including grade- or building-level data teams, vertical (across grade-level) teams, teacher- or leadership-led professional development, parent-teacher associations, parent-teacher conferences, or one of the many social networks available. Ideally, we, as educators, will be able to share what we are discovering about children's mathematical thinking, get insight and ideas from each other, and develop even better strategies to meet the needs of our students.

## Sample Classroom Activities

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**Activity #1: 10 Fruits in all**

*Context*

This activity is designed for a Kindergarten class or beginning first-grade class. With its open-ended nature, it can be particularly useful as a type of pre-assessment to help determine what students know about decomposing numbers, how they approach solving a problem of this nature, and their perseverance. The bansho instructional strategy will be used. Considering the developmental level of the students and the possibility of multiple solutions, this activity could run from one to three days.

*Objectives*

I can find combinations of 10 by using objects and/or drawings (K.OA.A.3).

I can represent my solution(s) with a drawing or equations (K.OA.A.3).

*Materials*

Math manipulatives (including cubes or counters that are green and red, if available), blank paper, writing utensils, ten-frames, blackboard/whiteboard/chart paper/butcher paper to be used as "bansho" (optional: a big bowl, cups, grapes, and cherries).

*Methods*

Before problem-solving: Gather students at the rug and tell them the following story, dramatically acting it out (with realia, if preferred) or requesting that students visualize: "Last night I made a fruit salad with grapes and cherries. I mixed them all up in a big bowl. I served them in special cups. Each cup holds exactly 10 pieces of fruit. How many grapes and how many cherries could be in each cup? Find as many combinations as you can." Tell the story two to three times. Have two student volunteers retell and/or act out the story problem using their own words. Request that the rest of the class listen carefully and show whether or not they agree with the retellings. Have students turn and tell the story problem using their own words. Display the story problem on the bansho. Have students share the information they already know (There are grapes and cherries. 10 pieces of fruit.) and record this information with visuals on the bansho. Ask students what we are trying to find (How many grapes and how many cherries?) and record on bansho.

During problem-solving: Provide students with access to manipulatives, blank paper, and writing utensils. On day 1, as students are working on solving the problem, mainly focus on observing and recording different strategies and solutions that the students are using, varying levels of concrete to abstract representations being used, as well as whether or not anyone has found multiple solutions. On days 2 and/or 3, continue observing and facilitating discussion, referring the students to the bansho to avoid misconceptions or encourage perseverance. For students who continue to struggle finding one accurate solution, consider providing a container with red (cherries) and green (grapes), as well as a cup to more concretely represent the story problem. Ten frames may also be useful if children are struggling to keep track of the "fruit" or record their solutions. To help facilitate students' mathematical thinking, ask: How are you representing how many grapes and how many cherries are in each cup? Are there any more combinations? How do you know? Can you find any patterns in different combinations? How will you know when you have found all the combinations? Would there be more or fewer combinations if my cup could hold exactly 5 pieces of fruit? 12 pieces of fruit?

After problem-solving: Gather students back at the rug with their representations. Select students to share out with the group based on your observations and the needs of the students. For day 1, a beneficial discussion would be focused on the strategies students are using to find combinations or the variety of ways in which students are representing their combinations. Work with students to sort representations on the bansho by strategy used or by method of representation. Facilitate discussion in a way that leads to the summary: There are lots of [strategies I can use/ways to represent my combinations], and they all can help us solve the problem! Revisit the bansho on days 2 and 3 to remind students of previous discussions and findings. On day 2, students will likely be ready to have a discussion on finding multiple solutions and making sure they are not the same as previously found combinations. By day 3, as a class you can work together to see if anyone (individually, in pairs, or the collective class) has been able to find all of the different combinations of 10 fruits. The teacher can show an additional way to represent the combinations by using a list of sums or a T-chart of grapes and cherries. If the majority of students are ready, you may choose to work with students to put the list of sums in order (1 grapes + 9 cherries, 2 grapes + 8 cherries, and so on) and show how this ordered list can help you organize your solutions to ensure there are no repeated solutions. Record any conclusions that the class comes to on the bansho and have these records accessible for future discussions or problems that may be related to this one.

*Ideas for further practice or extensions*

Have each student share out one of their combinations using a complete sentence (i.e. "4 grapes and 6 cherries make 10 fruits altogether"). Give them a cup with those quantities of fruit to eat for snack or bring home to share with their families. Provide students with a similar story but with different quantities or with different objects to see how/if the students are able to extend their mathematical understanding. Provide the same story but with known quantities of grapes, so that they have to solve for the missing part.

**Activity #2: Developing more efficient strategies**

*Context*

This activity is designed for a first-grade class nearing the middle of the school year. The students should already have had significant practice with solving addition and subtraction math problems within the context of story problems and cooperative games. At this point in the year, students should also be comfortable representing and communicating their mathematical thinking and strategies used. (Please note that the example in Figure 4 shows teacher-created representations based on students' strategies and mathematical thinking observed and/or orally communicated. Ideally, the representations used will be *student-created* .) Up until this point, all of the students' strategies have been validated and valued and it is important that they continue to be throughout this lesson. However, regardless of how efficient of a strategy a student is using, this lesson seeks to help students start to think more about their criteria for strategies they select and begin to aim for more efficient strategies. An adapted form of the bansho instructional strategy will be used, as well as embedded (but explicit) vocabulary instruction. This activity will likely take up one math lesson, but can easily be repeated with different problem types or as needed to reinforce more efficient addition strategies with similar or higher numbers.

*Objectives*

I can represent and solve addition problems more efficiently (1.OA.A, MP5, MP8).

*Materials*

2 large dice, addition games, math manipulatives, blank paper, writing utensils, blackboard/whiteboard/chart paper/butcher paper to be used as "adapted bansho."

*Methods*

Before problem-solving: Gather students at the rug and explain to them that today we will be working on solving and representing addition problems, first looking closely at the strategies we are using. Tell them to imagine that they are playing "Addition Bingo" (or any other familiar addition game) and roll the dice record the roll so that all of the students can see. Notice how a "dull roar" ensues as students automatically begin to use a variety of strategies to combine the two numbers. To avoid students blurting out their solutions, have students explain their strategy by whispering it into their hand and holding onto it so they don't forget what they did. Explain to students that everyone is going to carefully create detailed representations of the strategies that they used and then gather back on the rug.

During problem-solving: Provide students with access to a variety of manipulatives, blank paper, and writing utensils. Circulate and observe students' representations of their strategies. Encourage students to show important details. For example, if they "used their fingers," how did they use their fingers? Did they count each set of numbers individually and then count all of the fingers in those sets one by one? Or did they use their fingers to keep track as they counted on? Which number did they count on? If any students are still struggling to represent their strategy, give them the opportunity to orally explain and/or show their strategy and have a peer or teacher create a representation of their strategy.

After problem-solving: Gather students back at the rug with their representations. Select students to share out with the group based on your observations and the needs of the students. Work with students to sort their strategies either by tool used or by strategy used. Facilitate discussion and attention to details in representations in order to ensure that all strategies are sorted appropriately.

Explain to students that we are going to focus on how efficient our different strategies are. Have them repeat the word efficient 3 times. Explain that *efficient* is a describing word that means "performing or functioning in the best possible manner with the least waste of time and effort." [24] Also tell them the student-friendly definition: "When something is efficient, it doesn't take a long time to do." Explain to students that we will do some practice using the phrases "less efficient" and "more efficient." For example, normally if I have a question for a teacher on the 2nd floor, I pick up the phone, dial her number and ask her. Would it be "more efficient" or "less efficient" for me to write the question, put it in an envelope, write her address, put a stamp on the envelope, put it in the mailbox, wait for her to get it, read it, write me back, and receive her response? That would be "less efficient" because it would take a lot more time to get an answer. Continue with 2-3 more examples worded appropriately in which the students need to respond with "more efficient" or "less efficient": varying paths to walking to the music room, writing "Connecticut" or "CT," or putting away all pencils at once or one at a time.

As soon as most students have some understanding of the meaning of "efficient," explain how we will rearrange strategies based on how efficient they are. Have students help determine how to rearrange the strategies so that the resulting display is meaningful to them. (Figure 4 portrays the resulting display from a similar lesson.) As feasible, you may choose to have students go "head-to-head" with their peers or yourself using their strategies to solve the same problem to see which is more efficient. When the class has agreed upon how to sort strategies, be sure to reinforce that all of the strategies help us get to the same solution, but that some are more efficient than others. Encourage students to try out strategies that are more efficient for the rest of the lesson and future practice.

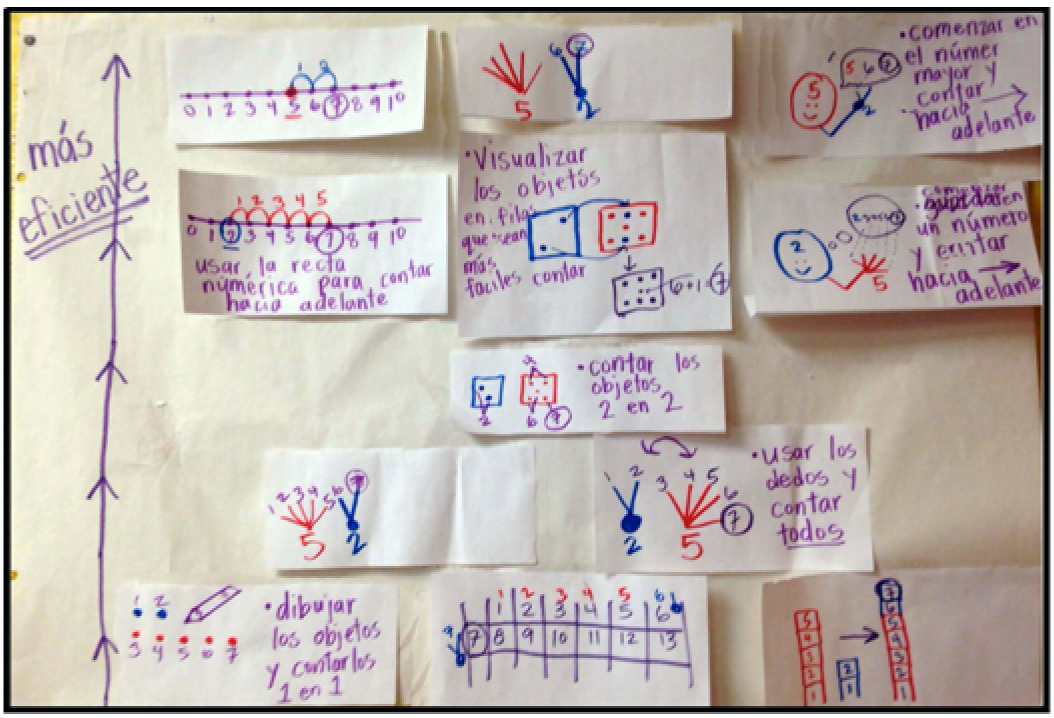



Figure 4: Example of sorting students' strategies by efficiency

**Activity #3: How many toy cars?**

*Context*

This activity is designed for a second-grade class in the last half of the school year. The students should have already had significant practice with solving one-step addition and subtraction story problems with numbers up to 100, as well as some exposure to two-step addition and subtraction math problems with smaller numbers. Students should also be comfortable representing and communicating their mathematical thinking and strategies used. Students should have had adequate experience selecting number sets that are not too easy or too difficult. An adapted form of bansho instructional strategy will be used, as well as the "just-right numbers" strategy to differentiate instruction. This activity will likely take up one math lesson, but can easily be replicated with different math problems and/or sets of numbers as needed to reinforce students' mathematical thinking.

*Objectives*

I can solve two-step word problems by using drawings and equations with a symbol for the unknown number to represent the problem (2.OA.A.1).

*Materials*

Math manipulatives, blank paper, writing utensils blackboard/whiteboard/chart paper/butcher paper to be used as "adapted bansho," a "just-right" story problem.

*Methods*

Before problem-solving: Gather the students at the rug and do a few quick warm-up problems that will help the students start thinking within the context of the story problem planned for the lesson: 1) Maria had 12 toy cars and 3 blocks in her backpack. How many toys did she have in her backpack in all? (no action, part-part-whole, whole unknown);

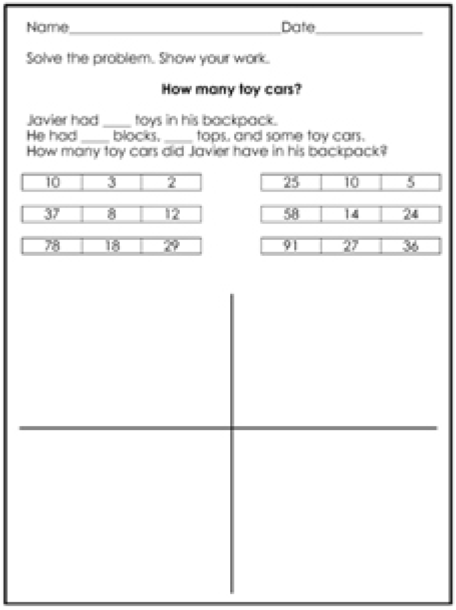
2) Jimmy had 18 toys in his backpack. There were 14 tops and some blocks. How many blocks does Jimmy have in his backpack? (no action, part-part-whole, part unknown);

3) Carla had 4 blocks, 3 tops and 7 toy cars in her backpack. How many toys did she have in all in her backpack? (no action, part-part-part-whole, whole unknown). Explain to them that today we will be working on solving and representing two-step problems. Present the following problem to students, by displaying it on the board: "Javier had 10 toys in his backpack. He had 3 blocks, 2 tops, and some toy cars. How many toy cars did Javier have in his backpack?" Have a couple of student volunteers read it out loud and others retell the story using their own words. Have students share out what is known and unknown in the story and record this on the board. If your class has worked with bar modeling to help organize information in story problems, and a student suggests it, you may wish to record:

| 10 toys | | |
|---|---|---|
| 3 blocks | 2 tops | ? toy cars |

If the majority of your class has started to successfully use and understand abstract representation, and a student suggests it, you may wish to record: 10 = 3 + 2 + ___

During problem-solving: Distribute "just-right" problem paper (see Figure 5) to students and remind them briefly that they should try to start with a problem that is not too hard, not too easy, but "just-right." Provide students with access to a variety of manipulatives, blank paper, and writing utensils. Circulate and observe as the students work to solve problems. Take note as to the strategies that they are using to solve the different problems. Some questions that can be asked to help facilitate the students' mathematical thinking include: What are you trying to find in this problem? How many blocks and tops are there in his backpack? How do you know? How can you prove it? Does your solution make sense? What if Javier miscounted and there were actually 99 toys in his backpack? How would your solution change? Are there more efficient strategies you could use to solve the problem? Is that the most efficient strategy for all of the problems? Why?


Name________________________Date____________

Solve the problem. Show your work.

**How many toy cars?**

Javier had ____ toys in his backpack.
He had ____ blocks, ____ tops, and some toy cars.
How many toy cars did Javier have in his backpack?

| | | | | | |
|---|---|---|---|---|---|
| 10 | 3 | 2 | 25 | 10 | 5 |
| 37 | 8 | 12 | 58 | 14 | 24 |
| 78 | 18 | 29 | 91 | 27 | 36 |

Figure 5: A "just-right" problem

After problem-solving: Gather students back at the rug with their papers and facilitate discussion based on what you observed students doing while solving the problem and based on the needs of your students. Some areas of focus may be: strategies used to organize a two-step problem, abstractly representing the problem, methods used to check your work and prove that your solution is correct. Sort and organize the students' representations based on the focus. Use "5 Productive Talk Moves" as needed to facilitate a rich discussion. Conclude by developing a summary of key concepts that the students collectively discovered and displaying them on the bansho. Allow students to record these key concepts in their notebooks or directly on their homework so that they can refer to it when doing additional practice for their homework.

## Resources

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### List of Materials for Classroom Use

- "Bansho": chart paper/butcher paper/blackboard/(interactive) whiteboard

- Paper: blank and graph paper of varying sizes, shapes and colors, notepads, etc.

- Writing utensils: pencils, crayons, colored pencils, markers, pens, clipboards, etc.

- Manipulatives: base 10 blocks, two-sided counters, 10 frames (blank and filled showing different quantities), snap cubes, unifix cubes, pattern blocks, color 1" tiles, beans (or other found items that can be used as counters), Cuisenaire rods, fraction bars, tangrams, etc.

- Other math tools: hundred chart, number line, calculators, clocks, etc.

- Materials for games: dice, decks of cards, etc.

- Technology to help record students' strategies: cameras, iPads, video cameras

- Anything else that you think could be used to promote mathematical thinking

## Appendix—Implementing District/State Standards

This unit includes strategies and activities that can be used in the K-2 classroom, as well as adapted for 3-12 classrooms, that address all of the Mathematical Practices (MP) outlined in the CCSS, particularly the following:

- MP1: Make sense of problems and persevere in solving them. The establishment of a mathematical community, a routine for solving math problems, as well as all of the activities described help to support this standard.

- MP4: Model with mathematics. All of the work throughout this unit provides students the opportunities to apply the mathematics they know and understand.

- MP5: Use appropriate tools strategically. Suggestions are made to provide students with a variety of tools when solving mathematical problems, as well as representing them. In particular, Activity #2 facilitates opportunities for students to evaluate the efficiency with their use of these tools when solving problems.

- MP8: Look for and express regularity in repeated reasoning. When students are provided with multiple opportunities to solve a variety of problems, they become increasingly familiar with general methods they can use. More specifically, Activity #2 allows for students to identify which of their methods may be viewed as viable shortcuts.

While the above mathematical practices can be connected to any of the Standards of Mathematical Content, the activities in this unit focus on using them with K-2 standards within the domain of "Operations & Algebraic Thinking." More specifically, the following standards are addressed:

- K.OA.A.3: Decompose numbers less than or equal to 10 into pairs in more than one way, e.g. by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5=2+3 and 5=4+1). Addressed in Activity #1.

- 1.OA.A, 2.OA.A: Represent and solve problems involving addition and subtraction. Addressed throughout unit, within routine for solving story problems, and in all 3 activities.

- 2.OA.A.1: Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem. Addressed in Activity #3. [25]

## Notes

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1. "Operations & Algebraic Thinking," *Common Core State Standards Initiative* .

2. Dr. Fong Ho Heong, Chelvi Ramakrishnan, and Bernice Lau Pui Wah, *Math in Focus* , T4-T10.

3. Common Core State Standards Initiative.

4. Mitchell Stephens, *The Rise and Fall of the Word,* 16.

5. Elizabeth Spelke as quoted on "Words," *Radio Lab* .

6. Susan Rich Sheridan, in "Scribbles," 18.

7. Ibid.

8. Ibid.

9. "Concrete-to-Representational-to-Abstract (C-R-A) Instruction."

10. Maulfry Worthington and Elizabeth Carruthers, "The Art of Children's Mathematics, 5.

11. Kristine Reed Woleck, "Listen to Their Pictures," 215-27; Carole Saundry and Cynthia Nicol, "Drawing as Problem-Solving," 57-63; Stephen P. Smith, "Representation in School Mathematics," 263-74.

12. Elizabeth Carruthers and Maulfry Worthington, *Children's Mathematics* , 87.

13. Ibid., 79-82.

14. Ibid., 149-50.

15. Bernard Bagnall, "Children's Mathematical Writing."

16. Ibid., 222-24.

17. "Maximizing Student Mathematical Learning in the Early Years," 4-5.

18. Chapin, O'Conner and Anderson as quoted in "Maximizing Student Mathematical Learning in the Early Years," Ibid., 5.

19. "Bansho (Board Writing)."

20. Ibid., 205.

21. Ibid.

22. Thomas P. Carpenter, Elizabeth Fennema, Megan Loef Franke, Linda Levi, and Susan B. Empson, *Children's Mathematics* .

23. Ibid., 7-12.

24. "Efficient," *Dictionary.com,* accessed July 1, 2014, http://dictionary.reference.com/browse/efficient.

25. Ibid.