Curriculum Units by Fellows of the Yale-New Haven Teachers Institute
1980 Volume VII: Problem Solving

Topology

Curriculum Unit 80.07.08
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Introduction

There are several facets to mathematical literacy and its teaching. One of the facets requires the ability to reason logically.

My aim in this unit is to develop the skill of logical reasoning through topology using classifying, patterning.

The topics chosen are presented in a way that requires a minimum mathematical background and maturity. It is not assumed that students who use this unit know how to solve even simple equations or that they have much acquaintance with geometric figures.

The unit is intended for 7th, 8th, 9th grade students for 4 weeks, 2 to 3 periods a week, ideally the last weeks of the school year.

The unit is in seven sections. These sections are explored with the help of investigations.

These investigations are an essential part of the unit and are particularly suitable for group work. The value of these investigations lies in the thought which students put into their efforts to seek solutions rather than in any results which may be obtained. A suggestion for' adding a group which is unable to make a start appears at the end of the comments on each investigation. Several of the investigations offer opportunities for discussing the use of algebra in stating general results.

(figure available in print form)

COURSE OUTLINE

1. Topological Transformations

A topological transformation is the changing of one shape into another without cutting, breaking, filling up holes or joining, for example, when we make a doughnut from plasticene or clay and transform it into a cup.

See figure 1.

Two shapes such that each is a topological transformation of the other are said to be equivalent. See figure 1 for example.

The result of applying a topological transformation of a CIRCLE is called a SIMPLE CLOSED CURVE. See figure 2 for example. Drawings (a), (b), and (c) are SIMPLE CLOSED CURVES.

2. Nodes and Networks

A point with at least one path leading from it is called a NODE. The order of the node is the number of paths leading from the node. In figure 3, for example, B is 4node, A is 1node, and C is 1 node.

A NETWORK is a collection of points and curves joining some of the points. See figure 3 for example.

3. Arcs and Regions

A line joining two nodes is an ARC. An area bounded by arcs is a REGION. The area outside a figure is also a region. See figure 3 for example. In this network there are 3 arcs, 3 nodes and 2 regions.

4. Traversable Networks or Graphs

A network or graph is said to be TRAVERSABLE if it can be drawn with one sweep of the pencil, without lifting the pencil from the paper, and without tracing the same arc twice. It is permitted to pass through nodes several times. See figure 4 for example. Network 5 is NONTRAVERSABLE.

5. Inside or Outside

A line segment that starts outside a simple closed curve and finishes inside, crosses (TOUCHING IS NOT COUNTED) in an ODD number of points. See figure 6 for example.

6. Coloring Regions

Regions of a network are colored so that regions with a common arc have different colors. (Regions with a common node may have the same color.) No one has yet found a plane design which needs more than four colors. If you think you have found a map which needs five colors, see if someone else can color it in four. In figure 7, for example, 3 colors are needed.

7. The Moebius Band and Other Surfaces

See figure 8. There is something remarkable about the drawing.

This BAND, called a MOEBIUS BAND, is named after one of the pioneers in the subject of topology, a German mathematician who was born in 1790 and wrote a paper about its properties.

You will discover that the band with a half twist has unusual properties.

(figure available in print form)

1. TOPOLOGICAL TRANSFORMATIONS

Provide each student with a copy of figure 9. Ask them to make:

1. (a) A list of the ways in which the drawings differ from one another. Example: Some drawings are made only from segments of straight lines, others are not.
(b) A list of the ways in which the drawings are alike. Example: Each drawing divides the page into the same number of cells or regions.
1. (a) and (b) These questions are intended for class discussion, but the discussion is sometimes more fruitful if it is preceded by short group discussion or individual consideration of the questions. There follows an example of such a discussion on (b).

Question: In what way are the figures alike?
Answer: They all have four regions.
Question: Regions?
Answer: They divide the page into four parts.
Answer: Five parts.
Question: Why five?
Answer: There's the outside too.
Question: Do all the drawings have five regions or parts?
Answer: Yes.
Answer: They are all joined up.
Question: What do you mean by joined up?
PAUSE
Question: Draw some figures that are not joined up.
Answer 1: See figure 10.
Answer 2: See figure 11.
Answer 3: See figure 12.
Answer 4: The ones in figure 9 have a boundary, these haven't.
Answer 5: They have a boundary and two lines cross inside.
Question: Will any of these do? See figure 13.
Answer: No.
Question: But two lines cross inside.
Answer: The lines must join the boundary at different points.
Question: Draw some other figures like the ones in figure 9.
(figure available in print form)

Students will probably express themselves in informal and imprecise language, but the discussion should bring out the following ideas:

1. Distance, angle and direction are not preserved.
2. Any one of the figures could be stretched and pulled into any one of the others if distance, angle and direction are ignored.
3. The number of lines meeting at a point and the order of points on a line do not change.

Topology is about points, lines and the figures they make, but length, area, curvature and angle can be altered as much as you wish.

(i) We call bending and stretching (BUT NOT TEARING OR JOINING) TOPOLOGICAL TRANSFORMATIONS.

(ii) Properties that are still true about a drawing or network after it has been transformed are called invariant because they do not change or vary. The distance between two points is not invariant.

(iii) Two curves such that each is a topological transformation of the other are said to be equivalent.

EXERCISE A

1. Which of the drawings in figure 14 are SIMPLE CLOSED CURVES? Provide each student a copy of figure 14 and add more diagrams. (USE OF CLAY OR PLASTICENE CAN BE FUN AND HELPFUL TO FIND CORRECT ANSWERS.)
2. Can the first curve of each pair in figure 15 be transformed topologically into the second curve? NOTE: ADD MORE DIAGRAMS.
3. Which of the curves in figure 16 are topologically equivalent to the straight line segment A B?
4. State which of the drawings in figure 17 are topologically equivalent to each other?
5. Which of the pairs of drawings in figure 18 are equivalent? If you think a pair is equivalent, copy the second member of the pair and mark on it a possible position for A, the image of A under the transformation. If more than one position is possible, mark all of them. You are allowed to turn the drawings over.
6. What can you turn a beetle into? The original insect and one suggestion is shown in figure 19.
7. State which of the shapes (i)(v) are topologically equivalent to each of the shapes (a), (b), (c) and (d). See figure 20.
8. Draw some topological transformations of each of the drawings in figure 21.

(figure available in print form)

(a) Look at figure 23. There are four paths from the point B. One leads to A; where do the others lead?
(b) How many paths are there from C?
(c) State the order of each node marked with a letter in figure 23.

EXERCISE B

1. Make a list of the letters which have among
others: NOTE: PROVIDE EACH STUDENT A COPY of 26
LETTERS.
(a) One 3node, (b) two 3nodes, (c) one 4node.
2. Why is it impossible to draw a figure with one 1node and no other nodes?
3. Which of the letters in question 1 are topological transformations of the letter C?
4. Which of the letters in question 1 are equivalent to the letter Y? How many nodes has each of these letters? What kind of nodes are they? NOTE: WE DO NOT COUNT 2NODES SINCE EVERY POINT OF AN ARC IS 2NODE WITH THE EXCEPTION OF ITS ENDS.
5. Complete the table for the networks in figure 24. NOTE: PROVIDE EACH STUDENT A COPY OF FIGURE 24 AND ADD 5 MORE DIAGRAMS.

INVESTIGATION 1

Draw, if possible, figures which have:

NETWORK	1NODE	3NODES	4NODES	5NODES	6NODES
1.	—	—	2	—	—
2.	—	4	1	—	—
3.	—	—	—	—	1
4.	4	—	1	—	—
5.	1	1	—	—	—

6.	—	1	1	1	—
7.	—	3	—	—	—
8.	—	2	1	—	—

(a) Make up some more examples of your own.

(b) When is it impossible to draw a figure?

(c) Try to find a rule for deciding whether or not a figure can be drawn.

3. ARCS AND REGIONS

Each of the networks in figure 25 has 5 nodes, 8 arcs and 5 regions. (DO NOT COUNT 2NODES.)

INVESTIGATION 2

(a) Complete the table showing the number of nodes (N), arcs (A) and regions (R) for the networks in figure 26. NOTE: Provide each student a copy of figure 26 and add 7 more networks.
(b) Ask your neighbor to check your results.
(c) Look for patterns in your table. Comment on these patterns. NOTE: It is possible that the result will be found in many forms, such as N + R = A2, NA + R = 2. This can be followed by a useful discussion as to whether or not these formulas are equivalent. Other results which have been noted by students include: (i) N + A + R is always even, (ii) A + RN is always even. It should be noted that the formula N + R = A+2 remains true when a finite number of 2 nodes are counted. In general, 2nodes may be ignored.

SUGGESTION: You can ask students to make an addition column for N+R.

4. TRAVERSABLE NETWORKS

EXERCISE C

1. (a) Show that networks in figure 27 are traversable and add three more traversable networks.
(b) Mark the point where you start (S) and the point where you finish(F).
2. Draw four traversable networks of your own, showing the start and finish.
3. Draw a traversable network with two 2nodes and no other nodes. Where can you start and finish? Can you start from more than one point? Can you draw a network with just two 2nodes which is not traversable?

(figure available in print form)

4. Which of the networks in figure 26 are traversable? You may have to try several starting points before you either succeed or decide that the network is not traversable. NOTE: Provide each student a copy of figure 26 and the following table.

Net-work	Total Number of Nodes	Number of Even-Nodes	Number of Odd-Nodes	Is Network Traversable?
1.	2	2	—	Yes
2.	4	—	4	

.

.

10.

Note: In figure 26 network 1: A is an even node because 4 arcs meet at A and B is also an even node. In network 2: A, B, C, and D are odd nodes because 3 arcs meet at each node.

INVESTIGATION 3

Investigate possible starting and finishing points in traversable networks. When is a network traversable?

SUGGESTION: Note odd and even nodes. Look carefully at answers to question 4.

INVESTIGATION 4

The drawings in figure 28 are not traversable.

a) What is the least number of strokes in which each can be drawn?
b) Can you find a rule for deciding how many strokes you need by just looking at a figure?
c) How many extra lines we must add in order to make each drawing in figure 28 traversable and add 5 more non traversable networks to the list.

5. INSIDE OR OUTSIDE

Figure 29 is called a SIMPLE CLOSED CURVE, "SIMPLE" because it is formed by one continuous line which does not cross itself. It is easy to see that the point marked A is inside the curve.

Figure 30 is also a simple closed curve, but now it is not at all easy to see if point A is inside or outside.

A Frenchman named Jordan discovered an easy way of being sure. Just draw a straight line from the point to the outside curve. If it crosses the curve an odd number of times then the point is INSIDE. If the line crosses the curve an even number of times then the point is OUTSIDE.

INVESTIGATION 5

> a) Figure 31 shows a simple closed curve and two points A and B. Which point is outside the curve? Can you reach B from A without crossing the curve?
> b) Figure 32 also shows a simple closed curve and four points A, B, C, D. Which of the four points are inside the curve? How can you tell?
> c) Draw other simple closed curves. Try to find a rule for deciding which points are inside your curves.

SUGGESTION: Ask students to draw several different SIMPLE CLOSED CURVES. Choose a point inside each curve and draw a straight line from the chosen point to the outside of the curve. Count how many times the line cuts the curve. Can you now discover a rule?

6. COLORING REGIONS

> a) Provide each student with a copy of figure 33 and add 5 more networks. Ask them to color it so that regions with a common arc have different colors. Try to use as few colors as possible. State the number of colors you need in each case
> b) Draw networks of your own which need: (i) Only two colors. (ii) At least three colors. (iii) At least four colors. Can your neighbor color any of your networks with fewer colors than you have used?
> c) Can you find a map that needs at least five colors?
> d) Draw a map with eight regions which needs only three colors.

EXERCISE D

1. Provide each student with a copy of figure 34 and add 5 more networks. Ask them to color them using as few colors as possible.
2. Design repeating patterns of your own which are suitable for: (a) a kitchen floor, (b) a book jacket, (c) a wallpaper. Color them using as few colors as Possible .

Note: In Figure 26 Network 1: A is an even node because 4 arcs meet at A and B is also an even node. In Network 2: A, B, C, and D are odd nodes because 3 arcs meet at each node.

EXERCISE E (MISCELLANEOUS)

1. Figure 35 shows a plan view of a small cottage. Can you start at A and walk through every door of the cottage exactly once? SUGGESTION: Make a topological drawing by making each room into a NODE and each door into an ARC. Can you start anywhere except at A and succeed in traversing the network?
2. The Koenigsberg Bridge Problem: In 1737 the Swiss mathematician Leonard Euler was working at the court of Catherine the Great of Russia. He was asked to solve the problem of the bridges of Koenigsberg. This town is now found on maps under the Russian name of Kaliningrad. There are two islands in the River Pregel which runs through the town. Seven bridges cross the river as shown in figure 36. The problem was this: Is it possible to take a walk which crosses each of the bridges once and once only? Euler solved the problem by changing it into a problem about the nodes of a network. See if you can solve the problem.
3. THE COLOR GAME: This is a game for two players. The first player draws a region. The second player colors it and draws a new region. The first player colors this region and adds a third. The game continues until one of the players is forced to use a fifth color; this player is the loser. Play this game with a neighbor.
4. The diagram 37 shows a simple ground floor plan of a house. Could you walk through this house so that you walked through each door once and once only? SUGGESTION: Make a topological drawing by making each room into a NODE and each door into an ARC.
5. GAME CALLED SPROUTS: This game was invented by two mathematicians at Cambridge University in England. This is a game for two players. Make three dots anywhere on a sheet of paper.
Each player in turn draws a line joining a dot either to itself or to another dot and places a new dot on this line. Some possible opening moves are shown in figure 38.

(figure available in print form)

(figure available in print form)

RULES OF "SPROUTS"

(i) No line may cross itself, cross another line, or pass through any dot.

(ii) No dot may have more than three lines leaving

(iii) When a line is drawn, a new dot must be chosen somewhere on it.

(iv) Each line must join two dots or else one dot to itself.

The last player able to move wins the game. A typical game is shown in figure 39.

A and B are the only dots which do not already have three lines leaving them and it is not possible to join A to B without crossing another line. So this game was won on the seventh move by the first player.

Play several games with a friend; Sprouts is not as simple as it looks!

Look carefully at some finished games. Regard them as networks. How many arcs are there in each of the networks? How many nodes? What is the order of each of these nodes? Can you explain why a game of sprouts must end after at most 8 moves?

It is possible to vary the game by starting with 2 dots. After how many moves must the game now end?

What happens if you start with 4 dots or 5 dots or ..?

Why is it unsatisfactory to start with only 1 dot?

7. THE MOEBIUS BAND AND OTHER SURFACES

EQUIPMENT: 3 strips of paper about 20 cm long and 3 cm wide, scissors, and Scotch tape. SUGGESTION: HOW TO INTRODUCE THIS TOPIC TO STUDENTS.

PART 1

Provide each student with a strip of paper.

1. Ask them to check on the number of faces and edges. If you put a finger on the edge you will notice that you can go right round and back to where you started without crossing a face. This shows that there is only one edge. To get from one face to another you have to cross an edge. This shows that there are two faces.
2. Take the next strip and join the ends together to form a loop. Check on the number of faces and edges now.
3. Hold the next piece by its ends, give one end a twist through two right angles and join the ends

together. You will find it harder to check on faces and edges this time.

Check on edges with your finger as before but make a pencil mark to show where you start. To check the faces it is advisable to shade a face with a pencil.

Provide each student a copy of figure 40 and table. Ask them to complete this table.

The band with a HALF TWIST is called a MOEBIUS BAND. It has only one side and one edge. The band's one-sidedness gives it a number of strange properties.

Some of these properties have been put to practical use. A rubber conveyor belt in the shape of a MOEBIUS BAND lasts longer since it has only one side instead of two. A continuous loop recording tape sealed in a cartridge will play twice as long if it has a twist in it The MOEBIUS BAND is even put to use in the design of electronic resistors!

EXPERIMENTS A:

Some of the properties of the MOEBIUS BAND are so unusual that they are hard to predict or even imagine unless you have discovered them yourself.

SUGGESTION: Provide each student with 3 strips of paper about 20 cm long and 3 cm wide.

Take one of the strips. Hold by its ends. Give one end a twist through 180 degrees. Join ends together with Scotch tape. Prepare two more bands like this.

1. Take one of the Moebius Bands. Draw a line with a pencil all round the middle of the band continuing until you come back to the same point from which you started. What happens?
This proves that the band has only one side since in drawing the line you never crossed over the edge.
Now cut the band along the line you have drawn. If you can guess the right answer you are a genius.
What is the result?
2. Take another Moebius Band and cut all the way round about a third of the distance from one edge continuing until you come back to the same point from which you started.
(a) What is the result?
(b) How do the loops compare in length?
(c) How do they compare in width?
3. Take third Moebius Band and this time cut along onefourth of the way from an edge until you come back to where you began.
(a) In what way is the result similar to the previous one?
(b) In what way is it different?
(c) Without trying it out, can you guess what the result would be if you cut around a Moebius Band one fifth of the way from an edge?
Provide each student with 2 strips of paper about 20 cm long and 3 cm wide. Give one end a twist

through 2 halftwists. Join ends together with Scotch tape. Prepare one more band like this.

4. Take one of the bands. Draw a line down the center of the band.

(a) How many sides does a band with 2 halftwists in it have?

(figure available in print form)

(b) Cut the band along the line. (c) Describe the result.

5. Take second band.

(a) Cut the band onethird of the way from an edge.

(b) Describe the result.

(c) Without trying it out, what do you think would be the result if you cut around a band containing 2 half twists onefourth of the way from an edge?

6. Take a strip 20 cm long and 3 cm wide. Give it 3 halftwists. Join ends together.

(a) Draw a line down the center of the band.

(b) How many sides does a band with 3 halftwists in it have?

(c) Cut the band along the line.

(d) Describe the result.

There are other interesting models to be made by twisting strips, for example, 3 halftwists and cutting onethird of the way from an edge, onefourth of the way from the edge. Using 4 halftwists and so on, but it soon becomes very involved.

PART 2

The double Moebius Band is made by placing two strips together, giving both a halftwist, and joining the ends together. See figure 41.

This appears to be just two bands, one on top of the other. Put your finger inside the bands and you can run it all the way round and return to the place you started. An ant crawling between the bands could go round and round forever as if it were walking between the surfaces of two separate bands where one was the ceiling and the other the floor. However, if the ant made a mark on the floor and circled until it reached the mark again it will need two complete circuits to reach that mark. It is not two bands at all and you can easily show this by holding one piece and shaking, and it becomes one large band. It is not so easy to put back again.

A great mathematician of our century, G. H. Hardy, said, "A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas."

ANNOTATED BIBLIOGRAPHY FOR TEACHERS

1. Mathematics: A Human Endeavour, by Harold R. Jacobs, W. H. Freman and Company. Chapter 10 should provide enough information to teach the topics of Moebius Strip and other Surfaces, Networks and Topological Transformations. Excellent illustrations with an easy tofollow text. I highly recommend this book for teachers.

2. Mathematics, by David Bergamini. A book in Time Life's "Life Science Library" Series, 1963. "Topology: The Mathematics of Distortion," pp. 176191. Excellent introduction to the ideas of Topological Transformations, Moebius Strip, Map Coloring, Noncircular Circles and Mazes with No Insides and The Bridges of Koenigsberg. Beautifully illustrated and inspiring.

3. Mathematical Snapshots, by Hugo Steinhaus, Oxford, 1969. Chapter 12, "Platonic Bodies, Crossing Bridges, Tying Knots, Coloring Maps and Combing Hair, pp. 252281. Good for general reading.

4. School Mathematics Project: Book B, Cambridge University Press. Chapter 11. Excellent introduction to Topological Transformations, Simple Closed Curves, Traversable Networks, Inside or Outside, Coloring Regions and the Koenigsberg Bridge problem with lots of examples and illustrations. A must for those who wish to teach topology.

5. School Mathematics Project Book 2, Cambridge University Press. Chapter I. Good introduction to Topological Maps, Topological Transformations, Simple Closed Curves, Traversable Networks and Coloring Regions with examples and illustrations.

6. Experiments in Mathematics, Pearcy and Lewis, Longmans.

Stage I: pp. 4647. Good introduction to the idea of "Inside or Outside" with examples and illustrations. Well worth reading, it deals with how complicated mazes are transformed into simple networks.

Stage 2: pp. 4647. Good introduction to the idea of "The Moebius Band and Other Surfaces." Good for general reading.

Stage 3: pp. 2021. Good introduction to "Networks and Routes" and solution to the problem of the Seven Bridges of Koenigsberg, pp.

4243. Excellent photographs and introduction to the idea of “Topological Transformations,” with an easytofollow text.