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# Teaching Word Problems

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Problem solving and word problems are universal concerns. Proficiency testing and math anxiety are issues of the day. When I discuss thin paper with students and adults, they express interest, they want to know how to solve word problems, and how to teach the solution of word problems. They tell about their struggles and their successes, many learned after returning to the task; others have yet to return. Their experiences and observations, however, give insight into our teaching technique.

Of course, some of the interest is a desire for a simple solution, a universal technique. This paper is not such an answer. I can only give reassurance. The process for solving word problems is the same as the processes one uses for reading and writing. Those who teach reading advise reading with questions in mind, reviewing what was read, and thinking out the consequences of what was read. Writing instructors advise listing the main points (an outline, perhaps), writing, and then rewriting. The situation with word problems is analogous.

We read the problems with some questions of our own: What is unknown? What is given? What are we looking for? What relationships exist between the unknowns and the knowns? Then we devise a plan, like the writer's outline. Then we execute it, getting our solution. Next we review what we have done so as to have greater understanding, as in reading, and to make the solution clearer, as in the rewriting.

This paper is aimed at the solution of the typical Algebra problem. It is a coach reminding us of the steps we should follow regularly. As one gains skill with a technique many steps are squeezed into a few, one forgets the struggle of learning and rushes off to the solution and then the next question.

Word problems are not a topic to be taught at one time and then forgotten. Word problems are a theme of the algebra course. They should be taught throughout the year. They motivate manipulative skills. They give a use for knowing $3x5+8x+6 = 11x + 1$; see pickanumber.

In summary the paper exists because word problems are seen as important but difficult. The paper is directed to algebra. As far as problem solving is a universal concern it has broad applicability.

The primary motivation for this paper is George Polya's book *How to Solve It* . What he talks about and demonstrates in his writing is what teachers want to see happen in their classrooms. Since he draws problems from all of mathematics' especially geometry, the book is not easily read by Algebra I students. However, I do consider it mandatory reading for adults, particularly those who teach the solution of word problems. Teachers should continually refer to it as they cover word problems, treat it as a manual, Just as when teaching oneself a skill. If we try to teach ourselves we read instructions and practice, then reread and practice some more and so on until mastery is achieved. Each time the reading is compared to the experience, giving more insight Problems that arise from practice send us back to the text for answers. Likewise when teaching word problems, try Polya's technique, then reread. His book reminds us of what we want to do. It is so easy to tell students solutions rather than leading them to solutions by questions.

Polya and other teachers try to put themselves in the students' position. This is a way to keep asking the questions the students should ask. However, how o we know the positions of students who do not voice their difficulties?

Two recent books give insight into the minds of students who have difficulty with math: *Mind over Math* by Stanley Kogelman and Joseph Warren, and *Overcoming Math Anxiety* by Sheila Tobias. Both books tell of student reactions to word problems. Some find any statement of student math difficulties as an indictment of math teachers. I see these books as case studies showing what goes on in the minds of some of our students, the ones who do not ask questions, who do not share their ideas. When reading these books, especially the quotes from anxious students in *Mind over Math* , see how Polya's plan should be applied to avoid the students' difficulties.

I believe that Polya's fourth step, "Looking Back" is the place to work on student anxieties. Explore the students' ideas. If an idea can be extended to lead into a solution follow that lead. Be hesitant to reject any student proposal. Rejection may lead to anxiety. Question the student further, get the student to reject the strategy. If a plan is "wrong" it indicates some misunderstanding; go after the misunderstanding.

I found these two books to be painful. How could students miss the point by so much? In their remarks some seemed to say that they partially listened, that they skipped the math part of any reading. All of them needed a system such as Polya's. Ironically these are adults who have had successin other fields where they follow a similar process, reading and writing.

## PickANumber

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Before doing word problems the student must appreciate the variable concept. One strategy to show students a use of the variable concept is the pick a number game. The class has writing materials, the teacher tells them to pick a number and then dictates some operations for the class to perform on their numbers. When the students are finished, the teacher asks each one for a result and quickly tells the value of the original number. Interest will be aroused by the difference in time taken by the students to get the second number and the speed at which the teacher gets all the original numbers. Students will want to know the trick, "How was it done?" If the teacher wants to avoid mental arithmetic the operations can be selected to give a common result for any initial number.

*Example* Pick a whole number bigger than two. Multiply it by three. Subtract five from that answer. Circle that answer. Take your number again, and multiply it by eight, and add six. Add the new answer to the circled answer. (Algebraically 3x5 + 8x + 6 = 11x + 1)

Then the teacher asks for the result and gives the first number. Students should be asked for suggestions on how this is done. Suggestions the teacher should be ready for are ones such as "You have them all worked out ahead of time." So be prepared to make up new examples. This activity will allow for arithmetic practice. Some numbers offered as results by students will be incorrect and the teacher will ask them to check their work.

Before the students become frustrated it is time to show how variables are used to solve the problem. I hear a dialogue such as this. "Problem says pick a number. We don't know what it is so let us use a letter in its place. (That is the variable concept.) Let x be the number. Next we are to multiply the number by 3 that gives 3x. Then we are to subtract 5 which we indicate by 3x5. We were told to circle it so its underlined. Now take the number again and multiply by 8 getting 8x. Next add 6 for a result of 8x+6. Finally, add the result to the circled answer. 3x5+8x+6= 11x+1. (Here is a use of a manipulative skill.) So what does all this mean? Well, if we took your original number multiplied it by eleven and then added one, we would get the same result as all those other operations give. So to get your number all that needs to be done is to reverse the last two steps: subtract one from your answer and divide by eleven."

I see this shown in two columns the words on one side the algebra on the other. Since one less than the student's result should be divisible by eleven, this is an opportunity to practice arithmetic.

## A Wordy Word Problem

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Many students who have difficulty with algebra do not see variables as a simplification. They see variables as adding to their problems not solving them. Let us try to solve a word problem using only words, no symbols. This method will be open to more than one attack. The purpose is to show the students the advantages of symbols over words.

| | |
|---|---|
| *Problem* | In 3 years Mike will be twice as old as he was 8 years ago. Now old is he now? |
| *Solution* | In 3 years Mike will be twice as old as he was 8 years ago. In 3 years he will be 11 years older than he was 8 years ago. That difference, 11, is his age eight years ago. So he is 19. |
| *Check* | If Mike is 19 now, he will be 22 in 3 years. Eight years ago he was 11. 22 is twice 11, so we are done. |

*Discussion* Each reader may have a different way of reaching the solution, including trial and error. We must pay very close attention to each step. Nothing is automatic. We must ask and answer the question, "Why is that true?" after each sentence. You may grant my solution is correct, but feel my argument is beside the point, coincidental, or even fallacious. It surely in a "word" problem) Now look at the problem using symbols.

Let a variable stand for what we are looking for. What are we looking for? Mike's age now. So let ~ = Mike's age now. What other ages does the problem mention? Mike's age 3 years from now, and Mike's age 8 years ago. What do we do to someone's age now to get their age 3 years from now? What do we do to someone's age now to get their age 8 years ago? Three years from now he will be older so add 3 to his present age. Eight years ago he was younger so subtract 8 from his present age. Let us show this in an organized fashion.

Let x = Mike's age now.

x+3 = Mike s age in three years.

x8 = Mike's age eight years ago.

In 3 years he will be twice as old as he was 8 Years ago.

x 2<br>+ = ( )<br>3 x<br>8

x + 3 2(x 8)

x +3 = 2x 16

19 = x

Same answer as before so the check remains the same.

Notice that the wordy solution requires more insight, while the symbolic solution requires more preparation. To solve the problem we translate it into symbols which make an equation for us to solve. We solve it using the properties of numbers.

The rhetorical method treats each problem as a unique special, case. The symbolic method allows us to see generalizations. In this case, age problems, we need remember only a few common sense ideas. When we talk of age in the future we add the years to the age now. When we talk of age in the past we subtract the years from the age now. Lastly, the difference between peoples ages remains constant throughout their lives.

## The Scheme

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This is the main part of the paper. Knowing what a variable is we are now in a position to solve a word problem. It the class already has tried solving word problems we are in the position of examining the steps we went through more carefully. George Polya puts forth the following scheme of steps.

I. Understand the Problem.
II. Devise a Plan.
III. Carry out the Plan.
IV. Examine the Solution. Look Back.

My experience is that one must be more explicit in giving the steps

I. Understand the Problem.
   1. Read the problem.
   2. What are we looking for?
   3. Use the answer to step 2 to introduce variables.

II. Devise a Plan.
   1. What relationships exist between the variables and the givers?
   2. Use the answer to step 1. to write an equation.

III. Carry out the Plan.
   1. Solve the equation.

IV. Examine the Solution. Look Back.

1. Read the problem again to see how the solution of the equation relates to the question. Sometimes the answer is Just yes of no, not a number.
2. Check your answer with the words of the problem, not your equation.

Polya's list is meant to be general, applicable to both proofs and numerical problems. My list is directed more to the standard algebra problems. Some comments are in order. My students often claim to be at a loss as to what to do, or how to start. Many act as if they should be able to read the problem once and immediately write the solution. They must be reassured that rereading is behavior. Also those students who skip numbers in their reading, see the math anxiety references, may be prepared to deal with them on a second reading. If the student expects to read the problem more than once, perhaps, the anxieties can be broken into manageable issues.

If students are still stumped for what to do next, they could write their reactions to the problem and make comments about it. The comments would then be starting points from which the instructor may ask leading questions. Also if students are attempting to deal with anxiety by avoiding word problems they may come to realize this by reading their own words.

We learn best from examples. Here are some with discussion of the scheme.

*Problem* Two trains are 500 miles apart. They start towards each other at the same time, one's speed is 20 miles per hour faster than the other. They meet in 4 hours. How fast is each going?

*Understand the Problem* . One aid is to draw a picture or a map.

(figure available in print form)

To understand the problem we must answer some questions. At the start the teacher may have to do all the questioning, with time it is expected that the students will do all the work on their own. The following dialogue is conceivable.

I: What does the picture show? S: Together the trains traveled 500 miles in 4 hours. I: What are we looking for? S: How fast was each train going? I: Do we have a relationship? S: I don't see any. I: You said the picture showed the distance the trains went and that we were looking for the speed of the trains. So, is there a relationship between speed and distance? S: Yes, distance equals speed times the time traveled. I: Time, did you mention that before? Do you know anything about the time? S: Yes, both trains went for 4 hours. I: Seems like you have a plan. S: You are rushing me. We have not introduced any variables.

Hopefully the reader gets the point. So let us introduce some variables.

Let x = speed in miles per hour of first train, 4x its distance in miles.

x +20 = speed of second train, 4(x +20) its distance.

*Equation* Together the trains went 500 miles.

| | |
|---|---|
| 4x + 4(x+20) = | 500 |
| 4x + 4x +80 = | 500 |
| 8x +80 = | 500 |
| 8x = | 500 80 |
| 8x = | 420 |
| x = | (420)/8 |
| x = | 52 1/2 mph |
| x+20 = | 72 1/2 mph. |

Examine the Solution *. This is the part that needs more emphasis in teaching. Is it clear? Can we answer more questions than those called for? Looking Back seems hard when reading Polya, but once started it can be an endless task. Well, the first obvious thing to do is to check the answer.*

| | | |
|---|---|---|
| Check: First train 52 1/2 mph for four hours goes | 210 miles Second train 72 1/2 mph for 4 hours goes | 290 miles |

Total 500 miles.

From the check we see that we could have answered the question: How far did each train go? Also, how much farther did the faster train go? The last question could have been answered without solving the problem for the speeds. If a train goes 20 mph faster than another for 4 hours it goes 4(20) = 80 miles farther than the slower train. As Polya points out the more questions we can answer the more convinced we can be. Another opportunity to lessen anxiety. When we are learning, reassurance is not superfluous.

We asked the question, “Is it clear?” Many students and teachers find it convenient to set the problem up in tabular form, using the key idea D = RT, distance equals rate times time as the organizing principle.

We still have the same variables: x is the rate of the first train, x+20 the rate of the second. So here is the table and its explanation.

(figure available in print form)

We have two trains so a row for each. We have literal expressions for the speeds so fill in the rate column, x and x+20. Both trains traveled for 4 hours so 4 goes into the T boxes. The D boxes are filled in by using the formula, the justification for the table. The first column equals the product of the second and third. The map still tells us to add the distances to get 500, this is our equation same as before.

Tables are intended to clarify our thinking to give us an organized way to attack the problem. If the student finds the idea more of a burden than a help, the student need not use it. The math anxiety people sounded as if they had been forced to use such techniques, only to become more confused. Perhaps, the formula was not stressed enough by the instructor. Even good ideas can be abused.

Consideration must be paid to the students’ styles of learning. This is where rereading Polya comes in. One must hesitate to offer a solution. At the same time the student who is floundering needs help.

The train problem leads into a class of problems: rate problems. The standard algebra problems of Mixture, Coin, Work, Solution, and Interest can all be thought of as rate problems, Just like Distance problems. In each case one quantity equals the product of two others. One of the factors is usually time. This relationship sets up a table for each of these problems. Let us see this in the following example of a work problem.

*Example* If one card sorter will sort 27,000 cards in an hour and another will sort 39,000 in an hour, how long will it take them to sort 55,000 working together?

*Understanding the problem* . What are we asked for? The time it takes the machines working together. Can we get an upper limit on the time? An estimate? Well if they both worked for an hour 27,000 + 39,000 = 66,000 cards would be sorted. So, we know the time has to be less than an hour. This example was introduced as showing work problems are rate problems, so let us calculate the number of cards sorted per minute by each machine.

Some discussion of rates is in order; Some students will find an appeal to arithmetic satisfying. Per means divide, so cards per minute means the number of card in a certain time interval divided by the number of minutes in the time interval. Other students will look to the equation C = RT, Cards equals rate of sorting times Time sorting, and solve for the rate.

Returning to the example, 27,000 cards in an hour means 27,000 cards in 60 minutes. 27,000/60 = 450 cpm; 39,000/60= 650 cpm. Now fill in the table. We are looking for the time the machines work together it is unknown so call it t.

| | Cards | = Rate | Time |
|---|---|---|---|
| Sorter I | 450t | 450 cpm | t min. |
| Sorter II | 650t | 650 cpm | t min. |
| Together | 55,000 | | |

Cards sorted by one machine plus cards sorted by other is 55,000.

| | |
|---|---|
| 450t + 650t = | 55,000 |
| 1100t = | 55,000 |
| t = | 55,000/1100 |
| t = | 50 minutes. |

Examine the Solution . *Check it. 450 cpm for 50 min. gives 22,500 cards. 650 cpm for 50 min. gives 32,500 cards. These amounts total to 55,000 cards as was required, we have the solution.*

Solution problems are a different type of rate problem. They do not involve time, but still we have one variable is the product of two others. The amount of solute equals the product of the percent concentration times the amount of solution. The percent concentration serves in the role of the rate.

When looking for a relationship between the variables, givens and unknowns, think of common sense general principles. Solution problems are examples where this type of thinking can be used to make an intimidating problem manageable. If I have two pails of water one contains 5 gallons and the other contains 4, how many gallons would I have in a third pail if I poured the first two into it? If you said nine you can do solution problems. Here is an example.

Example . *If 5 gallons of 20% alcohol is mixed with 4 gallons of 25% alcohol, what will be the percent concentration of the new mixture?*

*Understanding the Problem* . The previous discussion tells us with three quantities: amount of solute, alcohol; the percent concentration; and the amount of solution. Also the first quantity is the product of the other two. Recall your arithmetic, to multiply by a percent change it to a decimal. 20% = .20; 25% = .25.

| | Amount Alcohol | Percent concentration | Amount of Solution |
|---|---|---|---|
| First Solution | 1 gal. | 20% | 5 gal. |
| Second Solution | 1 gal. | 25% | 4 gal. |
| Mixture | 2 gal. | P | 9 gal. |

The amounts of solutions were given, the percent concentrations were given. Using the formula we calculate the amounts of alcohol present. Now we use our common sense notion, if 5 gallons are added to 4 gallons we get 9 gallons and if 1 gallon is added to 1 gallon we get 2 gallons. We are looking for the percent concentration of the mixture so call that P. Now use the relationship between the columns to write an equation and solve it.

2 = P(9)

2/9 = P

P = 22.22%

This problem was more of an arithmetic problem than an algebra problem. In fact, it is analogous to the question about average speed students see repeatedly on standardized tests.

*Example:* If a car travels 5 hours at a rate of 20 mph and another 4 hours at a rate of 25 mph, what is its average speed for the 9 hours? ‘

*Understanding the problem:* Many students find the average between 25 and 20 getting

22.5 mph. They do not understand the concept of average speed. The average speed is the total distance divided by the total time. So we must know how far the car went. We need our relationship: D= RT.

| | D | = R | . T |
|---|---|---|---|
| First part | 100 m | 20 mph | 5 hrs. |
| Second part | 100 m | 25 mph | 4 hrs. |
| Total trip | 200 m | R | 9 hrs. |

As in the previous example we do not know the rate so call it R. The relationship between the columns gives us our equation.

200 = R(9)

200/9 = R

R = 22.22 mph

Looking back *. The definition average speed is less than the common error “average”: 22.22 versus 22.50. While the problem may be considered arithmetic, we should still point it out to our students. Why should they miss easy problems?*

Let us conclude these examples with a more algebraic solution problem.

*Example:* How many gallons of pure alcohol must be added to 5 gallons of 20% alcohol solution to make a 50% alcohol solution?

*Understanding the Problem:* As in the previous solution problem, we know that the amount of alcohol equals the product—of the percent and the amount of solution. We are mixing two solutions to get a new solution so we are adding amounts. We want to know how much pure alcohol to add, the quantity, so let that be x. What is the percent alcohol of a pure quantity of alcohol? Pure means 100% alcohol. Now, fill in a table.

| | Amt. Alc. = | % Alc. | . Amt. Solution |
|---|---|---|---|
| First Solution | 1 gal | 20% = 0.20 | 5 gal. |
| Pure Alcohol | X | 100% = 1.0 | X |
| The Mixture | 1 + X | 50% = 0.50 | 5 + X |

As is the previous solution problem when we mix two solutions together we add the amounts, and we get our equation from the column relationship.

| | |
|---|---|
| 1 + X = | 0.50 ( 5 + X) |
| 100 + 100X = | 250 + 50 X |
| 50X = | 150 |
| X = | 3 gallons. |

Looking Back. *It 3 gallons of pure alcohol are added to 5 gallons of solution, the new mixture is 8 gallons. At the same time the 3 gallons of pure alcohol were added to the one gallon of alcohol that was already in the solution for a total of 4 gallons of alcohol. 4 gallons out of 8 is 50%. We have solved the problem.*

## Examining this Paper.

When teaching be hesitant to present a finished solution, look to the students for suggestions. Standard methods can become mechanical and mysterious to the anxious student. Stress the common sense motivation of any plan devised. Stress the use of variables.

When learning be prepared to reread and retry problems. Many of those who claimed skill in solving word problems mentioned returning to them after being confused in Algebra I. The skills one gains by growing and progressing in other subjects can aid one in solving word problems. Give yourself a chance.

This paper has been aimed at two audiences: instructors and students. It you are an instructor, read Polya. If you are a student, I hope this has been of aid. If you still think you need help in word problems go back and try the examples on your own. Ask more questions than I did. Find other references. Don't give up. Remember at the start of Algebra we may consider it to be arithmetic using letters like numbers. The variable concept.

## Students' Bibliography

There are not many collateral references for students with lower level reading skills. Perhaps they may be encouraged by the quality of the books to work through them. For the student working independently on word problems the number of references is large. Such students could even use text books.

Asimev Isaac, *Realm of Algebra* . Greenwich, CT: Fawcett, 1961. Teaches Algebra. Reminds student that Algebra is arithmetic. It gives the student the chance to see math outside of a text book.

Hogben Lancelot, *The Wonderful World of Mathematics* . Garden City, NY: Garden City Books, 1955. A pictorial history of mathematics from the caveman to modern times. The student will see many uses for mathematics, learn about the problems that inspired mathematics. This is most likely the only book for the slow reader. It will not teach the solution of word problems, but it will serve as a motivator for studying math.

Jacoby, Oswald with William H. Benson, *Mathematics for Pleasure* . Greenwich, CT: Fawcett, t962. A collection of problems. Some are brain teasers, the answer in the statement of the problem, the others are algebra. There is a discussion prior to each problem set. There are worked solutions to all the problems. A review not an introduction.

Sawyer, W.W., *Vision in Elementary Mathematics* . Baltimore, MD: Penquin Books, 1964. This book teaches algebra, it works from the premise that one learns more easily it one understands rather than memorizes. The book guides the student to discover math. Sawyer talks to the student as a friend, he gives pep talks.

## Teachers' Bibliography

To teach word problems one should appreciate the difficulty of the students. One way to gain this appreciation is to attempt problems that are not the standard curriculum. So look for puzzle books.

Ball, W.W. Rouse, *A Short Account of the History of Mathematics* . New York, NY: Dover, 1960 reprint 1908 edition. Histories may be used to show the students that the difficulties they may be having are analogues to the experiences of mathematicians. Ball discusses "Rhetorical" Algebra.

Ball, W.W. Rouse and H.S.M. Coxeter, *Mathematical Recreations and Essays* . Hong Kong: for University of Toronto Press, copyright Trinity College, Cambridge, 1974. A problem book. If puzzles or more difficult "PickaNumber" examples can be used here is one source.

Biggs, Edith, *Mathematics for Older Children* . New York, NY: Citation Press, 1972. Directed towards discovery teaching. Something to read as one is teaching word problems it will remind one to ask not to tell.

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