Curriculum Units by Fellows of the Yale-New Haven Teachers Institute
1980 Volume VII: Problem Solving

A Chronological History of ¹ with Developmental Activities in Problem Solving

Curriculum Unit 80.07.11
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Introduction :

The unit begins with a historical development of ¹ and proceeds with examples of activities to help students develop a deeper appreciation of the mathematical beauty and values of ¹.

Prerequisites :

Students should have at least a working knowledge of fractions and decimals, plus an intuitive understanding of the nomenclature, terminology, vocabulary, and formulae of basic geometry.

Level:

The historical part of this unit may be used in any grade level, preferably middle school. The activities part could be used in any middle school level, although it is recommended for average and above average groups working in small groups or independently. It also may be used as a motivational or summary part of a geometry unit or pre algebra course.

The length of time needed could be a week or two, depending on the amount of time you would spend on the history of ~r and doing all or some of the suggested activities.

Part I: Chronological History

There are a number of Egyptian papyri that somehow have survived the ravages of time over some three and a half thousand years. The most extensive one of a mathematical nature is a papyrus roll about one foot high and some eighteen feet long which is now in the British Museum. It had been bought in 1858 in a Nile resort town by Henry Rhind. It is often known as the Rhind Papyrus or as the Ahmes Papyrus in honor of the scribe by whose hand it had been copied in about 1650 BC The scribe tells us that the material is derived from a prototype from the Middle Kingdom of about 2000 to 1800 BC It is possible that some of this knowledge may have been handed down from Imhotep, the architect and physician to Pharaoh Zoser, who supervised the building of his pyramid about 5000 years ago.

The Egyptians were accurate in counting and measuring, and the pyramids exhibit such a high degree of precision in construction that legends have grown up around them. One such legend is that the ratio of the perimeter of the base of the Great Pyramid of Cheops to the height was set at 2^{1}.

The Egyptian rule for finding the area of a circle has been regarded as one of the outstanding achievements of the time. The scribe Ahmes assumed that the area of a circular field with a diameter of nine units is the same as the area of a square with a side of eight units. If we compare this assumption with the modern formula A = ^{1}r2 , we find the Egyptian rule to be equivalent to giving ^{1} a value of about 3 1/6.

(Refer to Activity I)

The knowledge indicated in the Egyptian papyrus is mostly of a practical nature, and calculation was the chief element in the problems. The Ahmes papyri may have been only a manual intended for students.

For more mathematical achievements one must look to the river valley known as Mesopotamia. In the Mesopotamian valley the area of a circle was found by taking three times the square of the radius, an accuracy that falls considerably below the Egyptian measure.

(Refer to Activity II)

The leading mathematician of the Hellenistic Age was Archimedes of Syracuse. Archimedes studied for a while at Alexandria under the students of Euclid, but he lived and died at Syracuse. Details of his life are scarce, but we have some information about him from Plutarch’s account of the life of Marcellus, the Roman general. During the Second Punic War the city of Syracuse was caught in the power struggle between Rome and Carthage. The city was besieged by the Romans during the years 214 to 212 BC Throughout the siege Archimedes invented ingenious war machines to keep the Romans at bay; catapults to hurl stones, ropes, pulleys, and hooks to raise and smash the Roman ships, and devices to set fire to the ships.

As for his approximate evaluation of the ratio of the circumference to the diameter of a circle Archimedes again showed his skill. Beginning with the inscribed regular hexagon, he computed the perimeters of polygons obtained by successively doubling the number of sides until one reached ninetysix sides. This is sometimes called the Archimedean algorithm. The result of the Archimedean computation on the circle was an approximation to the value of ^{1} expressed by the inequality:

3.140845 ^{1} C 3.142857

is a better estimate than those of the Egyptians and the Babylonians. This result was given in Proposition 3 of the treatise *On the Measurement — the Circle* , one of the most popular of the Archimedean works during the Medieval period.

Ultimately, Syracuse fell, and Archimedes, at seventy five years old, was slain by a Roman soldier, despite orders from Marcellus that his life be spared.

It should be noted that neither Archimedes nor any other Greek mathematician ever used our notation ^{1} for the ratio of the circumference to the diameter of a circle.

Another leading mathematician of the Hellenistic Age, Apollonius, wrote a work entitled *Quick Delivery* in which he calculated a closer approximation to ^{1} than that given by Archimedes, probably the value 3.1416.

Now Ptolemy's approximation of ^{1}, used in the *Almagest,* is the same as 377/120, which leads to a decimal equivalent of about 3.1416, a value that may have been given earlier by Apollonius.

Heron of Alexandria is best known in history of mathematics for the formula for the area of a triangle:

(figure available in print form)

where a, b, c are the sides and s is half the sum of these sides. Heron's formula was known to Archimedes. The area of a circle with diameter 14 is easily found by following Heron's instructions, "Multiply 212 by 154, add 841, take the square root and subtract 29, and divide by 11" equals a diameter of 14. He chose a specific case in which the sum of the circumference, diameter, and area of a circle, was 212.

The axiom of Eudoxus (like magnitudes must be compared; example, area cannot be compared to volume) would rule out such a problem for the three; circumference, diameter, and area, are unlike dimensions. From a numerical point of view however, the problem makes sense.

(Refer to Activity III)

The extent of Roman acquaintance with science may be judged from the de *architecture* of Vitruvius, written during the middle part of the Augustine Age. Marcus Vitruvius Pollio, the author, was interested in problems involving approximate measurements. The circumference of a wheel with a diameter of 4 feet is given by Vitruvius as 12.5 feet, implying a value of 3.125 for ^{1} This is not as good an approximation as that of Archimedes, but is respectable for Romans.

(Refer to Activity IV)

The civilizations of China and India are of far greater antiquity than those of Greece and Rome, although not older than those of the Egyptians and Mesopotamians. Perhaps the most influential of all Chinese mathematical books was the *Chuichang suanshu* or *Nine Chapters on the Mathematical Art* . This book includes 246 problems on surveying, agriculture, partnership, engineering, taxation, calculation, the solution of equations, and the properties of right triangles.

Now the area of the circle was found by taking three fourths the square of the diameter or onetwelfth the square of the circumference, a correct result if the value of ~ is 3.

(Refer to Activity V)

The search for accurate values were more persistent in China than elsewhere. Values such as 3.1547,V10, 92/29, and 142/45 were found. In the third century Liu Hui, an important commentator on the *Nine Chapters* , derived the figure 3.14 by use of a regular polygon of 96 sides and the approximation 3.14159 by considering a polygon of 3072 sides.

During the sixth century, there lived a Hindu mathematician, Aryabhata, whose best known work, written in 499 was the *Aryabhatiya* , a slim volume, covering astronomy and mathematics. One statement in the *Aryabhatiya* to which Hindu scholars have pointed with pride is as follows: Add 4 to 100, multiply by 8, and add 62,000. The result is approximately the circumference of a circle of which the diameter is 20,000. Here ^{1} is equivalent to 3.1416.

(Refer to Activity VI)

Also, Aryabhata used the value V10 for ^{1}, which appeared so frequently in India, that it sometimes is known as the Hindu value.

(Refer to Activity VII)

The Chinese fascination with the value of ^{1} reached its high point in the work of Tsu Ch'ungchih; his value was 355/113, approximately 3.1415929. In any case, his results were remarkable for that age, and it is fitting that today a landmark on the moon bears his name.

In the fifteenth century, AlKashi contributed to mathematics and astronomy. AlKashi delighted in long calculations, and was proud of his approximation for 2^1, 6.2831853071795865. No mathematician approached the accuracy of this computation until the late sixteenth century.

Most of Western Europe now was involved in mathematics, but the central and most magnificent figure in the transitions was a Frenchman, Francois Viete. Only Viete's leisure time was devoted to mathematics, yet he made contributions to arithmetic, algebra, trigonometry, and geometry. Viete worked out ^{1} correctly to ten significant figures, apparently unaware of alKashi's still better approximation. An exact expression was far more to be desired? thus Viete gave the first theoretically precise numerical expression for ^{1}, an infinite product that can be written as:

(figure available in print form)

In 1655 John Wallis proved his best known results, the infinite product:

$2/^1=$ (1/2)(3/2)(3/4)(5/4)(5/6)(7/6)

(Refer to Activity VIII)

Through manipulation of Wallis' product of $2/^1$, William Brouncker in 1658 was led to the expression:

(figure available in print form)

(Refer to Activity IX)

Gottfried Leibniz's name is usually attached to the infinite series:

$^1/4$ = 1/1 1/3 + 1/5 1/7 + 1/9 ...,

one of his first discoveries in mathematics. This series arose out of his quadrature (squaring) of the circle, and is only a special case of the arctangent expansion,

(Refer to Activity X)

It was Leonard Euler's adoption of the symbol '~d in 1737 and later in his many textbooks that made it widely known and used. Euler delighted in relationships between the theory of numbers and his rough and ready manipulations of infinite series. He found the infinite series:

¹ = 1 + 1/2 + 1/3 + 1/4 1/5 + 1/6 + 1/7 + 1/8 + 1/9 1/10 + ...

Here the sign of a term, after the first two, is determined as follows: If the denominator is a prime of form 4m 1, a minus sign is used, if the denominator is a prime of form 4m + 1, a plus sign is used, and if the denominator is a composite number, the sign indicated by the product of the signs of its components is used.

(Refer to Activity XI)

By the use of these and other series, the nature of or may be computed to any desired number of decimal places.

(Refer to Activity XII)

In 1761, Johann Lambert, showed that ¹ is irrational (cannot be expressed as the ratio of two natural numbers). Joseph Liouville, in 1844, proved the existence of transcendental numbers (numbers which cannot be expressed as the roots of algebraic equations with rational coefficients). Charles Hermite, in 1873, proved that e, the base of natural logarithms, was a transcendental number. Using this information, and Euler's theorem that erri + 1 = 0, the German mathematician, Ferdinand Lindemann, in 1882, proved that ¹ is transcendental.

¹ to twentyfive decimal places is:

3.1415926535897932384626433 ...

Here is a mnemonic device to memorizing a good approximation of ¹: How I want a drink, Tropicana of course, after the heavy lectures involving quantum mechanics. The number of letters in the words will provide the values for the successive digits in approximating ¹ to fourteen decimal places.

Part II: Developmental Activities

Activity I

The scribe Ahmes assumed that the area of a circular field with a diameter of nine units is the same as the area of a square with a side of eight units.

Have students calculate the area of both figures first using the Egyptian ¹. 3 1/6. Than ¹ = 3.14, as known today, and compare the results with the Egyptian's approximation.

Activity II

A Babylonian scribe found the area of a circle by taking three times the square of the radius.

Have students calculate the area of the circle first using the Babylonian method A = 3r2, and than by today's method of A = ¹r2.

Activity III

Heron chose a specific case in which the sum of the circumference, diameter, and area of a circle was 212. Heron's instructions "Multiply 212 by 154, add 841, take the square root and subtract 29 and divide by 11" will be a circle with diameter 14.

Have students read and following Heron's instructions to calculate the diameter. Is it 14?

Activity IV

The circumference of a wheel with a diameter of 4 feet is given by Vitruvius as 12.5 feet, implying a value of 3.125 for ¹.

Have students using the formula C = ¹d, calculate the value

Now have the students calculate the circumference of the wheel using ¹= 3.14 given a diameter of 4 feet.

Compare: Was Vitruvius off by a great amount?

Activity V

In the *Nine Chapters on the Mathematical Art* , the area of the circle was found by taking three fourths the square on the diameter or onetwelfth the square of the circumference.

Have students figure the value of ¹ using the following information:

A = (3/4)(d)2 = (1/12)(C)2

¹ should be about 3.

Activity VI

One statement in the *Aryabhatiya* to which Hindu scholars have pointed with pride is as follows: Add 4 to 100, multiply by 8, and add 62,000. The result is approximately the circumference of a circle of which the diameter is 20,000.

Have students follow the above instructions and calculate ¹.

Activity VII

V10= 3.162277... is sometimes known as the Hindu value.

Have students calculate areas of circles, and circumferences of circles using V10 for ¹.

Compare results with calculations done using 7 ~ 3.14.

Activity VIII

Using John Wallis' infinite product:

2/¹ = (1/2)(3/2)(3/4)(5/4)(5/6)(7/6) ....

have students calculate ¹ using hand calculators.

How far does one have to carry out the product to approach 3.14?

Activity IX

Using William Brouncker's expression:

(figure available in print form)

have students calculate ¹ using hand calculators,

How far does one have to carry out the expression to approach 3.14?

Activity X

Using Leibniz's infinite series:

¹/4 = 1/1 1/3 + 1/5 1/7 + 1/9 ....

have students calculate ¹ using hand calculators.

How far does one have to carry out the infinite series to approach 3.14?

Activity XI

Using Leonard Euler's infinite series:

¹ = 1 + 1/2 + 1/3 + 1/4 1/5 + 1/6 + 1/7 + 1/8 + 1/9 1/10 + ...

where the sign of a term, after the first two, is determined as follows: If the denominator is a prime of form 4m 1, a minus sign is used, if the denominator is a prime of form 4m + 1, a plus sign is used, and if the denominator is a composite number, the sign indicated by the product of the signs of its components is used.

Have students calculate ¹ using hand calculators.

How far does one have to carry out the infinite product to approach 3.14?

Activity XII

Have students using hand calculators choose one of these Fourier series (named after Joseph Fourier), to compute ¹ to any desired number of decimal places:

/2 = (2/1)(2/3)(4/3)(4/5)(6/5)(6/7) ...

¹2/8 = 1 + 1/9 + 1/25 + 1/49 +...

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