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Some Mathematical Principles of Architecture

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by Fred DiTallo

There is a relationship between mathematics and architecture. That relationship is, at times, a partnership where one draws equally upon the other. Sometimes it is a marriage, a coupling that is inseparable, and at other times it is a fierce competitor where old standards are unwilling to give way to new discoveries.

This relationship is the focus of my paper and I will examine only a few of the many intricate parts of architecture. It is aimed at high school geometry students. Yet this same material could be modified for any level math student whether higher or lower. It is designed to stimulate curiosity and lead to personal research beyond the content of the paper. Some sections are designed to expand upon the current topics being studied in a geometry class, and here I will deal with geometric proofs, constructions, the Golden Ratio, and the Fibonacci Series.

Through this seminar, I have realized that there is a major distinction between building and architecture. That difference seemed moot at first, yet further studies shed light upon it. The tradition of architecture reaches back to the ancient Greeks and beyond. Yet that Greek culture was the first western civilization to incorporate a wide variety of ideas and ideals and to personify them through their architecture. Mathematical concepts and traditions owe a great deal to this civilization. Euclid organized a great body of geometric data in his "Elements". The Pythagorean Theorem and conic sections grew here also and are still being taught in today's high schools.

Howard Roark, the hero of "The Fountainhead" would disagree strongly with this view of architecture, yet it is still a valid one. One that allows me to trace not only its development, but the development of mathematical concepts as well.

Architecture involves math, engineering, design, handling raw materials and much more. Yet, the sum is more than its parts. There are two large parts that I wish to examine. They are: The Tangible and The Intangible.

The Intangible aspects of architecture affect the emotions and other perceptions. Things such as size, color, shape, space and proportion, directly and indirectly engage the senses. For insights into this area I am indebted to our seminar leader Professor Kent Bloomer of the Yale School of Art and Architecture for showing me such information existed and to the 18th Century author Edmund Burke for his detailed analysis of this phenomenon.

The Tangible aspects of architecture deal with those mathematical principles that are involved in designing and constructing buildings. I will refer to geometric concepts mainly, but also involve other mathematical ideas such as the Golden Ratio and the Fibonacci Series. I am grateful to the works of Euclid, and to Leonardo of Pisa (Fibonacci), as well as to Harry Reid, Head of the Math Department at Lee High School, and to Ray Davie, a fellow math teacher, for their timely assistance.

Objectives:

To suggest possibilities, to challenge, to instruct. To suggest the possible uses of geometry outside of the classroom, and to communicate the fact that geometry is a living, breathing entity that can take on many forms and that it has existed for a very long time. To challenge the student to use what he or she learns in new ways. To instruct the student in the subject of geometry and architecture. While instruction perhaps must occur before application, I want to open up the student to the exciting possibilities first and thereby to motivate his learning.

Strategies:

Using a series of photos and geometric constructions, I will show how the concepts under discussion have been embodied by architecture. They are presented as a suggestion of the many possible variations that do exist, and that can exist.

The photos will refer both to the Tangible and Intangible qualities of architecture. The constructions will refer to basic geometry and how the figures are done using no measuring devices as are known to modern architecture.

The student will gain some insights into the applications of his math, and a historical appreciation of its development.

Lesson Plans:

Lesson Plan Number One:

Objective:

To create an awareness among students of the existence of geometric shapes in their environment.

Activity:

The teacher will ask the class to locate various basic geometric shapes in the classroom, such as a line, a square, a point, a rectangle, etc. A point could be located where two walls and the ceiling meet and so on.

Perhaps you can take them on a tour through the school itself, or even an outside tour to locate additional shapes. How about the ellipse in a football, or the parabola in an arch?

These activities could take from one to three class periods depending on student interest. You could also have the students draw free hand renderings of the various shapes they find.

Lesson Plan Number Two:

Objective:

Teach students how to construct geometric shapes.

Activity:

Referring to the section on constructions, have the students duplicate the given steps using only a compass and straightedge.

Go over how to construct these geometric figures:

(1.) A line that is perpendicular to another line.

(2.) Two parallel lines.

(3.) An Equilateral triangle.

(4.) A Scalene triangle.

(5.) An Isosceles triangle.

This activity could take from 2-5 class periods depending on your students' ability to duplicate instructions plus any additional explanations needed by the teacher.

Lesson Plan Number Three:

Objective:

To expand the students' use of rigor and logical sequence in solving geometry problems.

Activity:

Referring to the section on proofs, as well as your classroom textbooks, have the students do the proofs related to the constructions they have just done. The outline for these proofs is included within this paper.

This activity could last from 2-5 class periods depending on student ability to handle proofs at this point.

Lesson Plan Number Four:

Objective:

To allow the student to discover the Fibonacci Series and the Golden Ratio in his own environment.

Activity:

This is an outside project that will ask the student to discover one or more instances of a Fibonacci Series and a Golden Ratio outside of class.

Using the section on “Proportions and the Golden Ratio”, instruct the student in using these ideas. Then, offer suggestions as to how to find the Golden Ratio and do several examples for the class.

Now turn it over to the students and have them find at least one example of a Golden Ratio and one example of a Fibonacci Series on their own. You can make suggestions, such as the rows of points in a pineapple, shell fish, celery stalks, leaves on certain trees, etc., for the Fibonacci Series, and they will have to experiment on finding a Golden Ratio themselves. Lots of buildings have it.

Have the students write a report on what they did.

The time to teach this will be 1-3 class periods. The length of time for their assignment should not be long, 2-3 days at most.

The Intangible

The two categories of architecture that I wish to discuss are the Intangible and the Tangible.

By the intangible, I mean something not immediately discernible to your reasoning powers. Something that is not quite clear in its meaning or effect. Something that causes an emotional response before you realize it.

Architecture has a story to tell. Sometimes it’s a nice cheerful story, and other times it’s a scary ghost story. Let’s examine the differences. Starting with space, the building encloses the space, and creates an effect on the remaining space around it. How this is handled is fundamental to the profession of architecture.

Is there lots of light, wide corridors, spacious rooms inside. Is the facade of the building pleasing and inviting? Or, are the halls narrow and poorly lighted? Are the windows too small, resulting in poor lighting and stuffy overheated conditions? Is the facade drab and uninviting? How about entrances. Is it easy to get into the building, or are things forbidding?

The intangible in architecture isn’t really intangible. Someone made it that way on purpose to cause the response that it causes. It was thought about, measured, and calculated. Someone decided that the door should be small or large, and if the corridors should be wide or narrow. All of this falls into mathematics also. Perhaps more on the engineering side of it. Absorption or refraction of light, noise level, sharp angles, etc.

Let’s talk about the response to this as the intangible aspect. If you go into a government building to pay taxes or maybe to get a license, the halls may be narrow and dingy; the floors may be marble and echo every footstep; there may be very dim lighting; the doors in the corridors all look alike; you step into the office you think you need and no one’s there or maybe lots of people are working behind desks with their backs to you. Kind of creepy? Yes, but no accident.

In his book “An Inquiry Into The Origins Of The Sublime And The Beautiful” Edmund Burke goes into quite a lot of detail as to what things cause certain reactions.

By the sublime, Burke means those things which connote danger and instill a feeling of fear and terror. By the beautiful, he means those things which connote a sense of joy.

Some things that are related to the sublime are: roughness, hardness, darkness, unpleasant smells, unpleasant tastes (bitterness), silence, vastness, uninterrupted repetitions, largeness of size, strong contrasts (such as coming into a dark building from bright daylight), loud sounds, deformity, and unpleasant proportions.

Some of the characteristics related to the beautiful are: smoothness, light, softness, color, pleasant smells and fragrances, brightness, gradual variation, gradual change, delicateness, fragility, gracefulness, elegance, and congruency in proportions. These characteristics are also related to ideals which are considered beautiful. Justice, Wisdom, Virtue, Love and Truth. Thus, the architecture itself can be used to communicate any of these things. It might be the terror of a mansion in a ghost story, or the serenity of a church.

The Intangible in architecture is the emotional impact that a building can have on a person.

As an experiment visit some buildings around New Haven. How does being in the Greyhound Bus Terminal make you feel? How about the Top of the Park Restaurant? What are some of the differences experienced being at Teletrack or City Hall? How about a court of law or your favorite tavern?

The buildings themselves can affect you, and that is the intangible quality of architecture.

The Tangible

While the intangible is something that is aimed at an emotional response, the tangible involves the underlying laws of math and science that allows a building to be designed and constructed, things such as angles, length, width, height, arches, circles.

We will learn how the ancient Greeks designed all their structures and made all the geometric shapes and calculations using only the compass and straightedge. It is a great tribute to the Greek civilization that the average high school geometry text taught in today’s high school is based on Euclid’s “Elements” dating to 300 B.C.

While the total body of mathematical knowledge is vast when compared to the geometry of ancient Greece, it is still that geometry that serves as a foundation for higher math.

Let us look at a few elements of the Tangible side of architecture:

Proportion And The Golden Ratio

At various times in history, certain styles and certain proportions have been popular in architecture.

According to Rudolf Wittkower in “Architectural Principles in the Age of Humanism”, musical relationships, and harmonic proportions were popular during the Renaissance. The use of the octave, the fifth and the fourth were also predominant. Also, the human proportion was popular such as the head compared to the body was 1:8.

One notable proportion of western architecture is the Golden Ratio of the ancient Greeks. It was considered an extremely pleasing proportion and was calculated by dividing the length by the width of a building resulting in a ratio of 1:1.61.

Referring back to Burke for a moment, one characteristic of beauty is pleasing proportions. And beauty is related to high ideals. These ideals were not discovered by Burke, but only analyzed in great detail. The Greek culture at its zenith personified these high attributes in their enduring architecture.

Around 1202 A.D. an Italian merchant, named Leonardo of Pisa, (also known as Fibonacci), wrote a book called "Liber Abaci". In this book he presented a problem relating to the sequence in which rabbits multiply. He revealed a mathematical fact of nature that is still being studied today.

He discovered that rabbits multiply in the following manner: First one rabbit, then one more rabbit, two rabbits, three rabbits, five rabbits, eight rabbits, thirteen rabbits, and so on. To find your next number, add the two preceding numbers together. Since one has no preceding number, you still get one, but after that watch out.

1 + nothing = 1	3 + 2 = 5
1 + 1 = 2	5 + 3 = 8
1 + 2 = 3	8 + 5 = 13
	13 + 8 = 21
	etc.

In a clearer way:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 . . .

n, n, n1, n2, n3, n4, n5, n6, n7 . . .

0 + n = n, n + n = n1, n1 + n = n2, n1 + n2 = n3, n2 + n3 = n4, n3 + n4 = n5, n(k-2) + n(k-1) = nk

What he discovered occurs in many other instances of nature, such as the rows of points on a pineapple; the stalks on celery; the petals of flowers; the chambers in certain shell fish; in the leaves of a cherry tree; and in the sections of a sunflower.

Where the Fibonacci Series relates to the Golden Ratio is when you get up to the number 13 and divide it into the next succeeding Fibonacci number, you then get the Golden Ratio.

i.e. 21 Ö 13 = *1.61* 5
34 Ö 21 = *1.61* 9
55 Ö 34 = *1.61* 7
89 Ö 55 = *1.61* 8

We consistently get the Golden Ratio that the Greek mathematicians and architects discovered based on observation. They incorporated it into their noblest structures, and in the 13th Century another man was able to rediscover it and expand upon this natural phenomenon. This validates the advanced stage of Greek civilization in many respects, especially in math and science.

Let’s turn now to a study of geometry as the Greeks themselves used it and look at some constructions and proofs.

CONSTRUCTION OF PERPENDICULAR:

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1. Draw line AB.
2. Choose a point P not on AB.
3. Use P as the center of a circle of any radius that intersects AB at two points. Mark the points of intersection C, and D.
4. Now using C & D as centers of new circles, choose a radius greater than 1/2 CD.
5. Draw an arc from C & D that intersects at point E.
6. Draw line EP.
7. Line EP is now perpendicular to line AB.

(figure available in print form)

PARALLEL LINES

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1. Construct AB.
2. Construct external point P not on AB.
3. Draw a transversal through P that intersects AB at R.
4. Using compass, make R the center of a circle of radius less than . Draw an arc.
5. Keep the compass with the same opening and draw an intersecting arc from point Q.
6. Repeat at point P and then S.
7. SPD Å QRB.
8. Therefore CD||AB.

(figure available in print form)

Equilateral Triangle

1. Make line AB of any convenient length.

2. Open a compass to the length of AB. _

3. Use A as the center of a circle of radius AB.

4. Use B as the center of a second circle also of radius AB.

5. Where the two circles intersect at the top, label point C.

6. Draw lines AC and BC.

7. AC, AB, and BC are all radii of equal circles.

8. AC Å AB Å BC, therefore ’ ABC is equilateral.

(figure available in print form)

Construction of Isosceles Triangle

1. Draw line BC.
2. Open compass to length BC.
3. Use point B as the center of a circle of radius BC.
4. Repeat #3, but use C as the center.
5. Extend line BC to new length AD.
6. Open compass to this new length.
7. Make two new circles of radii AD using B and C as the center points.
8. Where the two larger circles intercept label it point E
9. Draw lines BE, CE, and BC.
10. BC is the radius of one smaller circle.
11. BE and CE are radii of congruent circles.
12. Therefore BE Å CE.
13. BE > BC. 14. ’ BCE is Isosceles per definition.

(figure available in print form)

SCALENE TRIANGLE

1. Draw line AB of any length.

2. Open compass to equal the length of AB.

3. Using A, and then B, as the centers, draw two circles of radius AB.

4. Put a point on the circumference of one circle and label it point C.

5. Draw radius AC.

6. Choose a point D on radius AC such that AD AC.

7. Draw a line from point B to point D making line BD.

8. AD AB.

9. BD > AB.

10. BD > AD.

11. Therefore, all lines are of different length, resulting in a Scalene Triangle.

(figure available in print form)

Proof of two parallel lines

Given: AB is a line, 5 Å $I t7 Prove: CD AB

Statement	Reason
EF is a transversal.	1. Definition of Transversal.
2. 5 = 7	2. Given
3. 6 Å 8	3. Opposite angles
4. 5 Å 3	4. Alternate Interior angles
5. 3 Å 1	5. Opposite angles
6. 5 Å 1	6. Substitution
7. CD |AB	7. Two lines are parallel when a Transversal forms congruent alternate interior angles

(figure available in print form)

Proof of Perpendicular Lines

Given: AB bisects CD Prove: AB CD

Statement	Reason
1. AB bisects CD	1. Given
2. CB Å BD	2. Definition of bisector
3. Construct lines AC and AD	3. Construction

4. ACÅAD	4. Definition of bisector
5. AB = AB	5. Reflexive property
6. ’ ABC Å’ BD	6. Side, side, side theorem
7. Choose a point E on AB	7. Construction

and construct CE and DE

8. CE Å DE	8. Definition of bisector
9. EBÅ EB	9. Reflexive property
10. ’ BCEÅ’BDE	10. Same as 6
11. AB CD	11. A line determined by two points equidistant from the ends of a line segment is the perpendicular bisector of the line segment.

(figure available in print form)

Proof of an Equilateral Triangle:

Given: AB Å BC, B Å C Prove: ’ABC is equilateral.

Statement	Reason
1. AB Å BC	1. Given
2. B Å C	2. Given
3. A Å C	3. Angles opposite equal sides are equal.
4. A Å B	4. Substitution
5. AC Å AB	5. Sides opposite equal angles are equal.
6. AC Å BC	6. Substitution
7. ’ ABC is equilateral	7. Definition of equilateral

(figure available in print form)

Isosocles Triangle

Given: ’ ABC with A Å B, CA Å CB Prove: ’ ABC is Isosocles

Statement	Reason
1. A Å B	1. Given
2. Draw BD such that B is bisected	2. Construction and definition of bisected
3. 1 Å 2	3. Definition of bisector
4. BD ÅBD	4. Reflexive property
5. ’ ABDÅ ’CBD	5. Side, angle, angle theorem
6. ABÅ BC	6. Corresponding sides of congruent angles are congruent.
7. ’ ABC is Isosocles	7. Definition of Isosocles

(figure available in print form)

Proof of a Scalene Triangle

Given: A > C, BC AC Prove: ’ ABC is Scalene

Statement	Reason
1. A > C	1. Given
2. BC AC	2. Given

3. C A	3. Based on 1
4. AB BC	4. Side opposite C is less than the side opposite A (based on corresponding sides opposite their angles)
5. AC > BC	5. Based on 2
6. ’ABC is Scalene	6. Definition of Scalene Triangle

(figure available in print form)

THIS PHOTO OF THE BACK COURTYARD OF WOOLSEY HALL HAS SEVERAL ELEMENTS. FIRST, SOME INTANGIBLE ASPECTS. THE SPACE SEEMS ARRANGED IN A PLEASANT MANNER. THERE ARE NO SHARP CONTRASTS TO SHOCK US. THERE ARE THE PLEASANT CURVING LINES OF THE WINDOWS WHILE THE COLUMNS PRESENT THE USE OF PERSPECTIVE ALLOW FOR GRADUAL CHANGE. THESE CHARACTERISTICS ARE TOWARD THE BEAUTIFUL AND GOING TO A CONCERT HALL WOULD PROBABLY BE A BEAUTIFUL EXPERIENCE. CONSIDERING THE COLUMNS ON THE LEFT AND ALSO ANOTHER VIEW OF THE SAISE COLUMNS, WHAT TANGIBLE QUALITIES CAN YOU FIND? WHAT SORT OF GEOMETRIC SHAPE ARE THE COLUMNS? WHAT KIND OF GEOMETRIC CHARACTERISTIC IS PRESENT IN ONE COLUMN NEXT TO ANOTHER? CAN YOU DISCERN ANYTHING ELSE?

(figure available in print form)

(figure available in print form)

IF YOU WERE ENTERING A BUILDING SURROUNDED BY A FACE LIKE THIS, WHAT EFFECT WOULD THAT HAVE ON YOU? ENTWINING SNAKES: THE UNINTERRUPTED REPETITION OF SPIKED POLES: THE SHARP CONTRAST OF A BLACK FENCE AGAINST A WHITE STONE BACKGROUND? UNCOMFORTABLE?

(figure available in print form)

ANOTHER EXAMPLE OF THE INTANGIBLE. THIS SPACE HAS A COMFORTING EFFECT. PLEASANT GREEN COLORS, BROWNS, AND OTHER EARTH TONES. PLENTY OF SPACE AROUND EACH BENCH AND THE GENTLE ARCH OF THE TREES.

(figure available in print form)

AN EXAMPLE OF THE INTANGIBLE. A QUIET CORNER. A GRADUAL CHANGE AS YOU COME THROUGH A COURTYARD INTO A WELL LIGHTED AREA. A GENTLENESS IS EXPERIENCED BECAUSE OF SHRUBS TAKING THE SHARP EDGE OFF THE BUILDING.

(figure available in print form)

EXAMPLE OF A CUBE—THE BEINECKE RACE BOOKS LIBRARY. THE CUBE ITSELF IS A BASIC GEOMETRIC SHAPE. CAN YOU FIND LARGE SQUARES OR RECTANGLES? WHAT OTHER SHAPES CAN YOU FIND

(figure available in print form)

HERE IS AN EXAMPLE OF USE OF THE GOLDEN SECTION OF GREEK ARCHITECTURE. THIS IS THE OLD POST OFFICE DOWNTOWN. ALLOWING FOR AN INACCURACY DUE TO PHOTO DISTORTION, WHEN YOU MEASURE FROM THE OUTER EDGE OF THE EXTREME LEFT HAND COLUMN ALL THE WAY TO THE OUTER EDGt OF THE EXTREME RIGHT HAND COLUMN, YOU GET 3 1/16 INCHES OR 3.0625 INCHES. THEN MEASURE FROM THE VERTEX OF THE FRIEZE TO THE BASE OF THE COLUMNS AND YOU GET 1 29/32 INCHES OR 1.9063. DIVIDE

3.06Z5 BY 1.9063 AND YOU CONE UP WITH 1.6065. THE GOLDEN SECTION IS 1.61. THIS WAS CONSIDERED THE MOST PLEASING PROPORTION IN GREEK CULTURE AND AN EXPRESSION OF BEAUTY. AS DISCUSSED EARLIER, EDMUND BURKE RELATES THE HIGH IDEALS OF WISDOM, TRUTH, JUSTICE, ETC., TO THE SENSE OF BEAUTY. GOVERNMENTS REPRESENTED SUCH HIGH IDEALS. THUS WE HAVE A POST OFFICE DESIGNED USING THE GOLDEN SECTION.

(figure available in print form)

THIS IS THE COURT HOUSE ON THE NEW HAVEN GREEN. THE—EDGE SECTION IS PRESENT IN THIS GOVERNMENT BUILDING. MEASURE THE WIDTH OF THE COLUMNS AT THEIR BASE AND YOU GET 4 7/8 IN. OR 4.875 IN. THEN MEASURE THE LENGTH OF THE COLUMNS THEMSELVES AND YOU GET AN EVEN 3 INCHES 4.875: 3.0 = 1.625. ALLOWING FOR PHOTO DISTORTION, YOU HAVE THE GOLDEN RATIO OF 1:1.61.

(figure available in print form)

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