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An Introduction to Mathematical Probability

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Introduction

The theory of mathematical probability is a little over three hundred years old. Since its discovery by Pascal and Fermat in the seventeenth century it has become an integral part of our everyday lives. At one time or another all of us have made decisions by taking chances, throwing dice, or drawing cards. Frequently we have made judgments based on the likelihood or probability that a certain event will happen. Developments in the theory and applications of probability range from simple informal activities to important fields as the physical sciences, genetics, the social sciences, economics, industry, engineering and insurance. Because of its widespread use, the theory of probability should be presented in high school mathematics classes.

In this unit of curriculum we will try to improve the students' understanding of the elementary ideas included in probability theory. The unit will clearly define important words and ideas. Formulas for solving problems will be presented. Permutations, combinations, and compound probability will be discussed. Applications of the theory to practical problems will be demonstrated in the unit. Following the explanation of each topic a set of practice exercises will be given. There are several basic objectives for this unit of study. Upon completion of the unit, the student will be able to:

> understand and appreciate the use of probability in everyday life.
> solve problems involving permutations and combinations.
> define basic terms used in mathematical probability.
> compute the probability that a certain event will happen.

The material developed here may be taught as a complete unit in algebra 3 classes, or parts of it may be extracted and taught at the following levels of instruction: (1) in seventh or eighth grade mathematics classes; (2) in high school applied mathematics classes; (3) in first year algebra or second year algebra classes; (4) in adult basic education classes.

Permutations

A *permutation* is an arrangement of a group of objects in a particular order. The four numbers 5, 6, 7, and 8 can be arranged in twenty-four different ways:

5678 5687 5768 5786 5867 5876
6785 6758 6587 6578 6857 6875
7568 7586 7856 7865 7658 7685
8567 8576 8657 8675 8756 8765

Each of these arrangements is a different permutation. To determine the total number of permutations that can be made from four digits using each one only once we indicate a space for each digit __ __ __ __. Any one of the four digits may be placed in the first space. Any of the three remaining digits may occupy the second space. There are two choices left for the third space, and the last digit is placed in the fourth space. The product of the number of choices for each space is the total number of permutations that can be made.

The product of the number 4 x 3 x 2 x 1 = 4! = 24

of choices for each space permutations.

The total number of permutations that can be formed from n objects using all of them without repetition is n! The

symbol n! is read *n factorial* . By definition, 0!= 1.

If n > 0, n! = 1 x 2 x 3 x . . . x (n-1) x n

nPn = n!   4P4 = 4! = 1 x 2 x 3 x 4 = 24

The symbol nPr represents the number of permutations that can be formed from n objects taken r at a time where r n.

The number of permutations of six objects taken three at a time is 6 x 5 x 4 = 120.

6P3 = 6 x 5 x 4 = 1 x 2 x 3 x 4 x 5 x 6 = ____6!____ = 120

3 x 2 x 1 (6-3)!

Hence nPr = __n!__

(n-r)!

Example 1 : How many different ways can five books be arranged on a shelf?

Solution : 5P5 = 5! = 1 x 2 x 3 x 4 x 5 = 120

Example 2 : How many two digit numbers can be made from the six digits 7, 2, 4, 5, 9, 3 if no digit can be used more than once?

Solution : 6P2 = __ 6__! = 1 x 2 x 3 x 4 x 5 x 6 = 30

1
x
2
(6-2)! x
3
x
4

Example 3 : How many integers of three places can be formed from the digits 5, 1, 8, 4 if repetition is allowed?

Solution : 4 x 4 x 4 = 64

Example 4 : How many even integers of four places can be formed from the digits 1, 2, 3, 4, 5?

Solution : 4 x 3 x 2 x 2 = 48

Example 5 : How many five digit telephone numbers can be made from the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9?

Solution : 9 x 9 x 9 x 9 x 9 = 59,049

Exercises :

1.) In how many ways can the offices of president, secretary and treasurer be filled from a group of nine people?
2.) In how many ways can five girls be arranged in a straight line?
3.) In how many ways can seven boys be arranged in a straight line if one particular boy is to be at the beginning of the line, one particular boy is to be in the middle of the line, and one particular boy is to be at the end of the line?
4.) How many integers between 10 and 100 can be formed by the digits 1, 2, 3, 4, 5 if no repetition is allowed? How many can be formed if repetition is allowed? 5.) How many odd-numbered integers can be formed by the digits 2, 3, 6, 5, 9, 8 if each digit may be used only once?
6.) In how many different ways can the letters of the word number be arranged if each arrangement begins with a vowel?
7.) A theater has five entrances. In how many ways can you enter and leave by a different entrance?
8.) In how many ways can you mail three letters in six letter boxes if no two are mailed in the same box?
9.) Milltown has eight grocery stores and six meat markets. In how many ways can you buy a pound of hot dogs and a bag of flour?
10.) Four people enter a bus in which there are six empty seats. In how many ways can the people be seated?

Permutations of objects that are not all different .

The four numbers 1, 6, 6, 3 can be arranged in 4! ways. Because two of the numbers are alike, several of the arrangements are the same and cannot be distinguished from others. Consider the digits in this way: 1, 6, 6*, 3. The following arrangements can be made:

16*63 1 6*36 166*3 1636* 136*6 1366*
6*631 6*613 6*136 6*163 6*316 6*361
516*3 6136* 6316* 636*1 66*13 66*31
316*6 3166* 36*16 36*61 3616* 366*1

Although twenty-four permutations have been formed, there are only twelve distinct arrangements. The number of distinct permutations of four objects when two are alike may be denoted by:

4! = 1 x 2 x 3 x 4 = _24_ = 12

2! 1 x 2 2

The number of distinct permutations of n objects of which s are alike, t are alike, etc. is

n! _

s! t! . . .

Example : How many different permutations can be made using all the letters of the word Connecticut?

Solution: The word Connecticut contains eleven letters including three C's, two N's, and two T's. The number of different permutations of these letters is

11! = 3! x 4 x 5 X 6 x 7 x 8 x 9 x 10 x 11

3! 2! 2! 3! x 2 x 2
= 1,663,200

Exercises

1.) How many different permutations can be made using all the letters of the word dinner?
2.) How many distinct permutations can be made using all the letters of the word (a) challenge (b) banana (c) staff (d) tuition (e) assassination (f) committee?
3.) How many different seven digit numbers can be made using all the seven digits 3, 3, 3, 4, 4, 5, 5?
4.) In how many ways can five nickels, three dimes, four pennies and a quarter be distributed among thirteen people so that each person may receive one coin?
5.) How many signals can be made by raising four red flags, two green flags, and one white flag on a pole at the same time?

Combinations

A *combination* is a group of objects in which the order or arrangement of the objects is disregarded. From the numbers 1, 2, 3, six different permutations can be formed. They are 123, 132, 231, 213, 312, 321. When the order of the digits is not considered, all six of these permutations make up one combination. The number of combinations of three objects, taken three at a time is one. 303 =

1. In general, the number of combinations of n objects taken n at a time is one. nCn =

1. When the number of combinations of n objects, taken n at a time, is multiplied by n!, the result equals nPn.

nPn

nC n x n! = nP n or nC n= n;

Similarly, the number of combinations that can be formed from n objects, taken r at a time is

nC r = nP y = n!

r! = (n—r)! r!

The number of combinations of n things, taken r at a time, is equal to the number of combinations of n things, taken n-r at a time, that is, nC r = nC nĐr.

nC nĐr = n! = n! = nC r

[n-(n-=r)] ! (n-r)! r! (n-r)!

Specifically, 40 C 38 = 40 C 2 = 40! = 40x39x38! = 780

2! (40-2)! 2x38!

In many cases this fact can be used to simplify computations.

Example 1 : In how many ways can a committee of four be chosen from ten people?

Solution: 10C 4 = 10! = 10 x 9 x 8 x 7 x 6! = 210

(10-4): 4! 6! x 4 x 3 x 2

Example 2: How many combinations can be made from seven objects, using them five at a time?

Solution: 7 C 5 = 7! = 7 x 6 x 5! = 42 = 21

(7-5)! 5! 2! 5! 2

Example 3: Evaluate 60 C 57.

Solution: 60 C 57 = 60 C 3 = 60! = 60!

(60-3)! 3! 57! 3!

= 60 x 59 x 58 x 57! = 34,220

57! x 1 x 2 x 3

Exercises

1.) Find the number of combinations of five objects taken from a group of nine objects.
2.) How many combinations of four items are there in a given set of six items?
3.) How many diagonals can be drawn in an octagon?
4.) In how many ways can seven questions out of ten be chosen on an examination?
5.) In how many ways can three books be chosen from five books?
6.) From a group of twelve ladies a committee of three is to be selected. In how many ways can this committee be formed with Mrs. Adams on the committee, but with Mrs. Jones excluded, if these two are part of the group of twelve?
7.) How many committees can be formed from a group of eight men, if one particular member of the group is to be included and two other members of the group are to be excluded from the committee?

Sample Space

To perform experiments scientists must select individuals or samples from a population. A random sample is one in which each member of the population has an equal chance of being selected. If selections favor certain individuals or events the sampling is said to be biased. The result of each experiment is called an event or outcome, and the set of all possible outcomes is known as a sample space.

Example 1: Each positive integer from one to five inclusive is written on a piece of paper, and the pieces of paper are shuffled. List a sample space for the outcome of drawing one piece of paper.

Solution: Any one of the five numbers is equally likely to be drawn. The sample space is (1,2,3,4,5).

Example 2: Two jelly beans are to be drawn from a jar known to contain only red jelly beans and green jelly beans. List a sample space for the result of the drawing.

Solution: Since the jelly beans must be red or green, a sample space is

{ (R,G), (R,R), (G,R), (G,G)}

Exercises

Complete each statement in exercises 1—4.

1.) A ___ ___ is a sample in which each member of a population is equally likely to be selected.

2.) A sample is ___ when certain individuals are favored in a selection.
3.) The result of an experiment is called an ___ or an ___
4.) The set of all possible outcomes is known as a ___.
5.) Each letter in the word flower is written on a card and the cards are shuffled. List a sample space for the outcome of drawing one card.
6.) Two balls are to be drawn successively from a bag known to contain only yellow balls and purple balls. List a sample space for the experiment.
7.) List a sample space that indicates all possible outcomes when two dice are thrown.
8.) List a sample space to show all possible outcomes when a family has three children.

Probability

Probability is the chance or likelihood that a certain event will happen. Mathematical probability is usually expressed as a ratio. If it is equally likely that an event may happen in h ways and fail to happen in f ways, where $h + f = n$, then the probability that the event will occur may be expressed as the ratio h/n, and the probability that it will fail to occur may be expressed as the ratio f/n. The fraction that denotes the number of favorable outcomes divided by the total number of possible outcomes represents the probability that an event will happen. The fraction that denotes the number of ways an event will fail to happen divided by the total number of possible outcomes represents the probability that an event will fail to happen.

$P(E)= h/n$ $P(\text{not } E) = f/n$

The sum of the probability that an event will happen, and the probability that it will fail to happen is 1.

$h/n + f/n = 1$

If it is certain that an event will happen, the probability of a favorable outcome is 1. If an event is certain to fail, the probability that it will happen is 0. The probability that an event will happen, P(E), ranges from 0 to 1.

0 P(E) 1

Example 1: What is the probability of throwing a four in one throw of a die?

Solution: The die may fall in any one of six ways. Only one of these will be “4”. The probability of throwing a four is 1/6.

Example 2: The faces of a cube are marked with the letters A, A, B, C, D, E. If the cube is tossed, what is the probability that an A will turn up?

Solution: The cube may turn up six different ways. There are two ways in which an A can turn up. The probability that an A will turn up is 2/6 or 1/3.

Example 3: A committee of three is to be chosen from ten girls. If Ann, Betty and Carol are among the group of ten girls, what is the probability that all three of them will be on the committee?

Solution: The total number of committees of three girls that can be chosen from ten girls is

$10\ _C 3 = 10! = 10 \times 9 \times 8 \times 7! = 120$

(10-3! 3! = 7! x 3 x 2

Ann, Betty, and Carol form one of these selections. The probability that the committee will consist of Ann, Betty and Carol is 1/120.

Exercises

1.) A bag contains 24 balls. Five of the balls are red, four are green, seven are blue, and eight are yellow. What is the probability that a ball picked at random will be (a) red? (b) green? (c) blue? (d) yellow?
2.) There are twenty-eight students in a class. Sixteen are girls, and twelve are boys. Find the probability that a student selected at random will be a girl.
3.) Find the probability that a number selected at random from the set of numbers 5, 6, 7, 10, 12, 14, 17, 21, 28, 30 will be divisible by 7.
4.) If you select a letter at random from the alphabet, what is the probability that it will be a consonant?
5.) If a number is selected at random from the set of numbers 1, 3, 17, 25, 71, what is the probability that the number is (a) an odd digit? (b) an even digit? (c) divisible by 3? (d) a prime number? (e) a composite number?
6.) If two dice are thrown, what is the probability of getting a sum of eight?
7.) Four marbles are drawn at random from a bag containing five orange marbles and seven brown marbles. What is the probability that (a) all four marbles are orange? (b) all four marbles are brown?
8.) If six cards are drawn at random from a deck of 52 cards, what is the probability that they are all spades?
9.) If a coin is thrown, what is the probability that it will turn up “tails”?
10.) In Hillcross High School there are 300 freshmen, 280 sophomores, 275 juniors, and 256 seniors. What is the probability that a student selected at random will be (a) a freshman? (b) a sophomore? (c) a junior? (d) a senior?

Mutually Exclusive Events

Mutually exclusive events are two or more events that cannot occur simultaneously. If one die is thrown and comes up three, it cannot come up six or any other number at the same time. If a coin is tossed and comes up tails, it cannot come up heads on the same toss. If a person weighs 125 pounds, he cannot have any other weight simultaneously.

The probability of one or the other of two mutually exclusive events happening is the sum of the separate probabilities of these events. If X and Y represent two mutually exclusive events

P(X or Y) = P(X) + P(Y)

This is known as the addition theorem and may be extended to any number of mutually eXclusive events.

Example 1: If a bag contains four blue marbles, six yellow marbles, and five green marbles, what is the probability that in one drawing a person will pick either a blue marble or a green marble?

Solution: There are fifteen marbles in the bag. The probability that a blue marble will be selected is 4/15. The probability that a green marble will be drawn is 5/15 or 1/3.

P(B or G)—P(B) + P(G) = 4/15 + 5/15 = 9/15 = 3/5

The probability that either a blue marble or a green marble will be drawn is 3/5.

Example 2: If a die is thrown, what is the probability that either a two or a six will come up?

Solution: The die can come up any one of six ways. The probability that a two will come up is 1/6. The probability that a six will come up is 1/6.

P(2 or 6) = P(2) + P(6) = 1/6 + 1/6 = 2/6 = 1/3

The probability that either a two or a six will come up is 1/3.

Exercises:

1.) Are the following pairs of events mutually exclusive?
- a.) Living in New Haven and working in New York.
- b.) Being a freshman and being a junior in high school.
- c.) Being a professor and being an author of a book.
- d.) Drawing a red card and drawing the ace of spades.
- e.) Drawing a face card and drawing the six of hearts from a normal deck of cards.

2.) If the probabilities that Joan, Beverly and Evelyn will be elected secretary of a ski club are 1/8, 2/5, and 1/3 respectively, find the probability that one of the three will be elected.

3.) If the probabilities that John and Harry will be valedictorian of a high school class are 1/4 and 3/7 respectively, what is the probability that either John or Harry will be valedictorian?

4.) Chris and Janet are among twenty girls who enter a tennis tournament. What is the probability that either one of these two girls will win the tournament?

5.) In a drawer are six white gloves, four black gloves, and eight brown gloves. If a glove is picked at random, what is the probability that it will be either white or brown?

Independent Events

Two or more events are independent if the occurrence of one event does not affect the occurrence of any of the others. If one coin is selected at random from a boX containing dimes and quarters and is replaced, the result of this selection would not affect the result of a second selection. These two drawings are independent events.

The probability that two independent events will both happen is the product of the separate probabilities. If X and Y are independent events, the probability that both X and Y will happen may be found by the formula

P(X and Y) = P(X) X P(Y)

Example: If a die is thrown twice, what is the probability that a five will come up on the first throw, and a three will come up on the second throw?

Solution: The probability that a five will come up the first time is 1/6. The probability that a three will come up the second time is 1/6.

P(5 and 3) = P(5) x P(3) = 1/6 x 1/6 = 1/36

The probability that both events will happen is 1/36.

To determine the probability of event X or event Y happening, when the two events are not mutually exclusive:

1.) Find the sum of the separate probabilities.

2.) From this sum subtract the probability that both events will occur.

P(X or Y) = P(X) + P(Y)—P(X and Y)

Example 1: When a card is drawn at random from a normal deck of 52 cards, what is the probability that it will be either an ace or a spade?

Solution: The probability of drawing an ace from the deck is 4/52.

The probability of drawing a spade is 13/52. The probability of drawing the ace of spades is 1/52.

The probability of drawing either an ace or a spade is:

P(ace) + P(spade)—P(ace of spades)

4/52 + 13/52—1/52 = 16/52 = 4/13

Example 2: If two dice are thrown, what is the probability that one of them will come up less than five?

Solution: The dice can come up the following ways:

11 21 31 41 51 61
12 22 32 42 52 62
13 23 33 43 53 63
14 24 34 44 54 64
15 25 35 45 55 65
16 26 36 46 56 66

The probability that the first die will come up less than 5 is 24/36.

The probability that the second die will come up less than 5 is 24/36.

The probability that both dice will come up less than 5 is 16/36.

The probability that one of them will come up less than 5 is

24/36 + 24/36—16/36 = 32/36 = 8/9

Exercises:

1.) If the probabilities that Mary and Sue will receive awards in a contest are 3/5 and 1/3 respectively, what is the probability that one or the other will receive an award?
2.) If five coins are tossed, what is the probability that all five coins will turn up heads?
3.) Find the probability that a person will throw 4, 8, and 10 on the first, second, and third tosses of a pair of dice.
4.) If two dice are thrown, what is the probability that one of them will come up greater than four?
5.) A bag contains six white balls, four green balls, and three brown balls. If three balls are drawn, one at a time, and the ball is replaced after each drawing, what is the probability that the balls drawn will be green, white and brown?
6.) A box contains four spools of black thread, six spools of brown thread, and ten spools of white thread. A spool is drawn, replaced, then a second spool is drawn. What is the probability that either a black or a brown spool is drawn?
7.) A card is drawn from a standard deck of 52 cards, replaced, and a second card is drawn. What is the probability that both cards are tens?

8.) The probability that Joe will solve a certain problem is 3/5, that Jane will solve it is 5/6, and that Sam will solve it is 1/4, What is the probability that Joe and Jane will solve it, and Sam will not solve it?

Dependent Events

Two events are dependent if the occurrence of one of the events affects the probability of the occurrence of the other. Consider a box that contains seven white, five green, and four blue pieces of chalk. If two pieces of chalk are drawn from the box, and the first piece is not replaced before the second piece is drawn, the outcome of the first selection affects the outcome of the second drawing. The probability of drawing a piece of blue chalk the first time is 4/16 or 1/4. If a piece of blue chalk is drawn first, then three of the remaining pieces of chalk are blue, and the probability of getting a piece of blue chalk on the second draw is 3/15 or 1/5.

The probability that two dependent events will happen may be expressed as follows: If pl is the probability that event X will happen and, after X has happened, p2 is the probability that event Y will happen, then the probability that the events will occur in the order X,Y is p1p2, the product of their respective probabilities.

Exercises:

1.) A bag contains five green marbles, four yellow marbles, and nine white marbles. If two marbles are drawn in succession, and the first marble is not replaced before the second is drawn, what is the probability that:

a.) the second marble is yellow, if the first marble drawn is green?
b.) the second marble is white, if the first marble drawn is yellow?
c.) both marbles are green?
d.) both marbles are yellow?
e.) both marbles are white?

2.) A boX contains ten slips of paper. Three slips are marked with the letter G, two slips are marked M, and five slips are marked K. If two slips of paper are drawn in succession, and the first is not replaced before the second is drawn, what is the probability that:

a.) the first slip is marked G, and the second is marked K?
b.) the first slip is marked G, and the second is marked H?
c.) the first is marked H, and the second is marked G?
d.) the first is marked K, and the second is marked G?

e.) the first is marked H, and the second is marked K?

f.) the first slip is marked K, and the second slip is marked M?

g.) both slips are marked G?

h.) both slips are marked M?

i.) both slips are marked K?

3.) The probability that Mr. Smith will be elected president is 5/7, and if he is elected, the probability that he will appoint Mr. Jones attorney general is 2/3. Find the probability that Mr. Jones will be attorney general.

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