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# Sole Geometry Activities

Curriculum Unit 87.05.04
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This year's seminar "Human Natures Biology, and Social Structure" sent me to many sources. I first read popular science magazines and science columns in the newspapers to have an overview of the topic. After that I went to more formal articles in *Scientific American* and *Science* .

I read about the structure of DNA and the genetic code looking for applications of mathematics. Places where the author acted as if everyone knew what he was talking about, or places where the author was using interesting mathematical concepts not usually covered in school. Two of the "everyone knows" topics were solid figures and symmetry. An interesting topic not usually in the curriculum was knots. It appears that the chromosomes tie themselves into knots. I would like to look at knots, but I would first have to learn more about the topic and then determine how to show it to my students so there would be something for them to do. I do not want to just impart facts to my students. Math is doing. I am at the point where I can see what I need to do and that I can do it, but I can not see it getting done soon enough to have time to write it into this unit.

The mathematics community regrets that students do not visualize three dimensions well. Solid geometry was once taught in high schools, but the press of topics lead to its loss with the thought that proofs in three space are just plane geometry arguments applied to planes of the solid figures. While that is true, how many times have we given thought provoking questions to our students, ones requiring the application of old stuff to new situations, only to be told: "We never had that."? Let us expose the students to as many experiences as possible. Geometry is the natural place to work on visualizing solids. If I am doing something for exposure, in an informal way, the best time to do that is the beginning of the year. A chance for the students to do something that is interesting and entertaining.

I see this unit as an opportunity to work out ideas that I have had for some time, but have not had time to explore. Ideas I have seen that look good but I have not gotten around to implementing. Such as the solid figures used as the covers of the 1985 issues of *The Mathematics Teacher* , or the group theory explained in the December 1985 issue of *Scientific American* .

When teaching Geometry, I want the students to see the structure, to learn what a mathematical system is. If the teacher or the book is too concerned with mathematical rigor, however, the students miss the point. Euclid's rigor is adequate for the beginning student.

Constructions are an opportunity to show the process. Start with constructions as an informal geometry class. Introduce definitions, show some "proofs", give the students a "hands on activity".

The mathematicians who have pointed out Euclid's deficiencies in rigor are correct mathematically, but not pedagogically. Postulates and axioms are supposed to be self-evident truths. So why would we prove something if it is self-evident? (Just to show that we can, is too sophisticated for the true beginner.)

So I want to make a chain of constructions that "prove" some important theorems of geometry. This will be done before we start following a text book. I want the students to become acclimated to the teacher, to doing homework, to keeping a note book and to thinking, before we start the official text book. Students need to be taught that they are responsible for topics that are not in the book or not in the section they are working on.

In the first week of geometry classes I give the etymology of geometry, some history of geometry, the flooding of the Nile, collection and discovery of rules about geometric figures, Euclid and the organization of the facts into a chain from easiest propositions to the hardest propositions. Proofs are a way to make geometry easier, to cut down on memorization. Students do not see geometry that way, they see it as too much memorization. Of course, they may be saying, " I don't want to work." My hope is to show how the process fits together so the students have an overall view of the project. So they will see what we are trying to do.

These are the constructions we will do.

1. Construct a line segment congruent to a given line segment
2. Construct a line segment congruent to a multiple of a given line segment.
3. Construct a triangle given three line segments.
4. Construct a triangle whose sides are in a given ratio, ( *e.g.* a 5,7,9 triangle).
5. Construct an equilateral triangle given one of its sides.
6. Construct an angle congruent to a given angle.
7. Construct an angle equal to the sum of two given angles.
8. Bisect a given angle.
9. Construct a line perpendicular to a given line.
10. Construct the perpendicular bisector of a given line segment.
11. Construct a square.
12. Construct a hexagon.
13. Construct a pentagon.
14. Construct a perpendicular to a line from a point not on the line.

These are the constructions needed to make the regular polyhedrons. There are others that I have required, construct parallel lines, for example, but I will do them when we cover those topics in the text. I see constructions as a topic to be recalled throughout the year.

The constructions of the regular polyhedrons will not be the first chain of arguments for the students. These construction activities are an opportunity to organize the informal geometric facts the students already know, and to give them those facts they missed. I will be asking questions such as “What do we need to be able to draw a circle?”, and “What determines a line?” These questions will lead to the statements of postulates that will end up in the postulate section of our note books.

When we are doing construction four I will ask the students if there are certain numbers that can not be used for the lengths of the sides of a triangle. This to solicit the triangular inequality.

While we are doing these constructions they are not going to “work exactly.” This will be a chance to talk about proof. It will also be a chance to mention symmetry as a way to check figures. For example, how would a loftsman check his batten for straightness? How would a carpenter check that his square is ninety degrees? How many symmetries are there for a regular triangle, a square, a hexagon, a pentagon and then also the regular polyhedrons?

The first major chain of argument will be the discussion of what is a degree and how many degrees there are in various angles. Many students feel that if the arms of an angle are lengthened then the angle is larger. Time spent early in the year on this concept will be well rewarded throughout the year. One activity to teach the meaning of angle size is to give the students angles cut out of colored paper and ask the students to find how big each angle is without using a protractor. Copy the angle until the copies total 3600. When the copies do not total 3600, give two numbers between which the angle must be.

Do this after construction eight and before construction nine. Draw a circle with center 0. How many degrees are there in a circle? Draw diameter AB. How many degrees are there in angle AOB? If you reflected the figure about AB what would happen? Would the figure look any different? So what can we say about the two angles on either side of diameter AB? So there are how many degrees in a straight angle? Now bisect angle AOB to get angle AOC. How many degrees are there in angle AOC? You have just done construction nine.

The construction of a regular pentagon is not usually found in high school geometry books. Here is a construction of a regular pentagon:

1. Draw circle 0 with a convenient radius. (The pentagon will be inscribed in circle 0.)
2. Construct diameters AB and CD perpendicular to each other.
3. Construct M the midpoint of OB
4. Draw circle M with radius CM to intersect AO at P.
5. Draw circle C with radius CP to cross arc AC at S. Chord CS is one of the sides of your regular pentagon.

In what follows I attempt to have a dialogue with the reader. I believe you are giving the correct answers, the answers I expect. When you see: (Answer 1) it means I have appended an answer to the question at the back of the unit so you can compare your expectations with mine. You should also draw the illustrations of what is being described. Remember to read math by checking the claims of the author. By leaving some answers out, it allows me to hand out copies of sections of the unit to students so they may work independently or prepare for future class discussions. The construction of a solid is not an activity that can be fit into the time restraints of a classroom. Students will need both time and patience to use their manual dexterity to make the solids.

Let us put what we have learned together to do something "big". What if I asked you to make models of all the possible regular polyhedrons? What would you need to know to be able to solve the problem? Make a list of questions. Here is mine.

> What is a regular polyhedron?
> How many degrees are there about a point?
> What is a regular polygon?
> What is a triangle, a square, a pentagon, a hexagon?
> How many degrees in each interior angle of a regular, triangle, quadrilateral, pentagon, and hexagon? (Answer 1)

If you know the answers to those questions you can tell how many regular polyhedrons are possible. The job was to make models of *all* the possible polyhedrons, so we want to know when we are done, we do not want to forget any.

When you said there are 360 degrees about a point you really meant what? Consider a cube it is made up of six squares. Look at a corner of a square, it is 90 degrees. There are three squares at each corner of the cube. Three times ninety is 270 not 360. What did you mean?

So if we are going to enclose space the angles around a vertex must add to less than 360 degrees. So let us make a list of what is possible. One angle of a regular triangle is 60 degrees, three 60 degree angles make 180, four 60 degree angles make 240, five 60 degree angles make 300, but six 60 degree angles make 360, too big. So we have three possible regular polyhedrons using triangular faces: 3, 4, or 5 triangular faces at a vertex.

Let us now look at the square case. Three 90 degree angles add to 270, but four 90 degree angles add to 360, so we have one possible regular polyhedron using square faces.

The pentagon is the next case. One angle of a regular pentagon is 108 degrees. Three 108 degree angles make 324, but four 108 degree angles make 432 degrees, which is bigger than 360, so we have one possible regular polyhedron using pentagonal faces.

Next comes the hexagonal case. One angle of a regular hexagon is 120 degrees. Three such angles make 360 degrees so we have no solid. Since the number of degrees in one interior angle of a regular polygon increases as the number of sides increases all the other cases will also fail.

We have seen there are, possibly, three regular polyhedrons with triangular faces, one with square faces and one with pentagonal faces. So there is a possibility of, at most, five regular polyhedrons. How can we stop saying “possibly”? Do they really exist? If we make them, there will be no question of their existence. Here is how to construct them. You will want to use paper or cardboard that you can cut out. I have had best results with paper, even though it is fragile the angles are much more precise.

Construct an equilateral triangle. Find the midpoint of each side. Draw line segments to connect each of the midpoints. Do you see four congruent equilateral triangles?

They are the faces of your polyhedron. Cut out the large triangle fold up on the lines. There is your figure. It is called a tetrahedron.

Construct a square. Using each side of the square as a base construct equilateral triangles extending out to the exterior of the square. Cut out the design, cutting along the outside boundary, the interior lines are fold lines. Fold up the triangles. Is this a regular polyhedron? No, one of the faces is a square. This is called a pyramid. Make another one with the same size base. Now place your two pyramids square to square. You now have a regular polyhedron. It is called an octahedron, it has eight faces. Design your own method to make a regular octahedron making a network as in the case of the tetrahedron, one piece no squares.

Here comes the big one. Lets see if I can tell you how to make it without giving you any illustrations. Pick a length for the side of your triangular face, call it s. Construct AB=5s. Now on each of the segments of AB construct an equilateral triangle, both above and below AB. Join the vertices of the triangles below AB to form line segment CD parallel to AB (have D close to B). Construct equilateral triangle DBE. Is line segment CE divided into 5 congruent segments? So use those segments to construct 5 equilateral triangles below CE. So cut out the network. Fold along the lines all folds in the same direction. Point A matches with point B, point C matches with point E, the 5 vertices above AB form one point and the 5 vertices below CD form another. (Answer 2)

If everything worked, you have a polyhedron of twenty faces, it is called an icosahedron.

Next comes the cube. It has six square faces and is called the regular hexahedron. Make one on your own.

Here is a question. Make two congruent regular tetrahedrons. Place them so two faces match. You now have a solid of six faces. Why is this not called a regular hexahedron? It is a hexahedron, but not regular. Why? (Answer 1)

The next case requires the construction of a pentagon.

Construct a pentagon. On each edge of the first pentagon construct another pentagon. You now have a star of six pentagons one in the center surrounded by five more. Cut out the network and fold up the pentagons to get a “basket”.

Do the same again to make a “lid”. Yes, put the lid on the basket.

This solid has 12 faces and is called the regular dodecahedron.

Instead of making a basket and a lid one can cause the two parts to pop up into the dodecahedron. Here is how. Place the stars on top of each other so only one star is seen. Turn one 36 degrees so you now see a ten pointed star. Take an elastic band place it under the points of the bottom star and over the points of the top star. You must press down on the center of the stars. Once the elastic is set, lift the pressure from the center, the elastic will serve as an equator pressing the points in and raising up the "north pole." See H. Steinhaus *Mathematical Snapshots* for a picture.

The idea of constructing six individual pentagons in a net will not be accurate. It is possible to construct one large pentagon and then divide it into the star of six. Construct the large pentagon, draw its diagonals, the pentagon in the center is the center of the star. Use the length of the side of the center pentagon as the side length for the others. The points of the large pentagon are the points of the pentagons of the star. Put your pattern over another piece of cardboard and cut two stars at once. It is possible by only drawing diagonals not to use your compass once the large pentagon has been made. See if you can do it. (Answer 3)

So we have made the five regular polyhedrons. Remember we proved there are only five regular polyhedrons, but there are polyhedrons that are not regular. How can we make polyhedrons? Do they have names? What do they look like?

If you use the dictionary to find the definitions of the polyhedrons, you will find that the prefix is the Greek number word for the number of faces and the suffix, hedron, is the Greek word for seat. So the polyhedron names are analogous to the polygon names. An icosahedron has twenty faces. We have no idea what kind of faces they are, or if they are all the same type, not by the name alone. If we are told it is a regular icosahedron then, yes, the faces are all equilateral triangles.

One way to make polyhedrons is to start with a polygon and a point not in the plane of the polygon. Connect the point to the vertices of the polygon, we now have a pyramid. If we make another copy and match the bases to form one solid we have a double pyramid.

Another method to make polyhedrons is to start with two parallel regular n-gons, with the sides of the polygons also parallel and draw line segments from each vertex of the one polygon to the corresponding vertices of the other polygon. This is called a prism. If the quadrilateral faces are rectangles the prism is called a rectangular prism. If the polygons started out with the vertices of one over the sides of the other, we could draw two line segments from each vertex, one to each endpoint of the side below, resulting in a band of triangular faces. This is known as an antiprism.

The previous two paragraphs tell us there are two other names for the regular octahedron: a double pyramid with a square base, or an antiprism with triangular bases. To see the double pyramid, hold the octahedron so it rests on one of its vertices with the opposite vertex above it. To see the antiprism, rest the octahedron on a triangular face. Notice the top face is a triangle parallel to the base with its vertices over the edges of the base. The "vertical" sides of the solid are triangles. Yes, a word game.

Polyhedrons are much more complicated than polygons. With polygons if you know the number of sides, all you have left to know is if the polygon is regular. With polyhedrons you need to know what kind of polygons are being used as faces, or if only one kind of polygon is being used, and how many polygons come together at each vertex and if it is the same number at each vertex. Some order can be given to this if we make restrictions on how to build the polyhedrons.

If we call for the use of only one type of regular polygon and the same number at each vertex we get the five regular polyhedrons, also known as platonic solids because the philosopher Plato used them in his theories.

If we use more than one type of regular polygon but require the vertices to be congruent we get the thirteen semiregular polyhedrons also known as the archimedean solids because Archimedes is credited with the first description of them. The stitching on present day soccer balls describes one archimedean solid, the truncated icosahedron. (Answer 4)

If we call for the use of equilateral triangles as faces but allow noncongruent vertices we get eight deltahedrons. The name deltahedron comes from the Greek capital letter delta which is shaped like an equilateral triangle.

This order still leaves an infinity of possible polyhedrons. All the polyhedrons mentioned so far have been convex. You can rest them on a plane and all the figure will be to one side of the plane. Concave solids would have faces that we could not place on the table or if we used imaginary planes then parts of the solid would be on both sides of the plane. Some examples of concave polyhedrons are star polyhedrons. They make great ornaments. See the bibliography for picture books.

Symmetry is a property of geometric figures. It leads to the concept of a group. Groups can be used by crystallographers when they determine the structures of molecules. Mathematicians use groups in many ways. One use is to show that certain geometric constructions are impossible with compass and straightedge alone. Not that no one has found a way, but rather, no one can ever find a way.

When Euclidean constructions are performed we use only a compass and a straightedge. The compass draws circles and the straightedge draws straight lines, it has no marks on it, it can not be used to measure lengths. This means the points of intersection of circles and lines will be the solutions of certain equations. If we know that the solution needed for a construction is not of the type we get from circles and lines then we know the construction can not be done by compass and straightedge alone.

Symmetry can be used to check the accuracy of regular geometric figures. How can we check to see if an equilateral triangle is in fact equilateral? We already used our compass and straightedge so lets find some other technique. Assume the triangle we want to check has been cut out and each of its verticles has been labeled. Put matching labels on both sides of the cutout. Now trace the triangle on a piece of paper and label that triangle to match the cutout. Now the question is, "How many ways can the cutout be moved so that it matches the tracing?" The answer is the number of symmetries of an equilateral triangle. So a symmetry of the triangle means that we could pick up the triangle and put it back down again and not notice any change if the labels were missing. Have you thought about an answer to the question?

Let us work it out. Set up the configuration, the cutout inside the tracing, labels matching. We could rotate the cutout 120 degrees counterclockwise. Record the position of the moved triangle.

*(figure available in print form)*

Go back to the matching situation. We could rotate it 240 degrees counterclockwise.

*(figure available in print form)*

We could leave angle A fixed, but flip the triangle about the bisector of angle A. We could do the same with angles B and C.

*(figure available in print form)*

Do we have all the symmetries?

What would happen if we flipped on angle A and then rotated 120 degrees counterclockwise?

*(figure available in print form)*

It is just as if we flipped on angle B in the first place. This is the concept of composition of functions or composition of symmetries. If we do one operation and then follow it with another operation it is the same as if we did just some third operation in the first place. So how many symmetries of an equilateral triangle are there? What operation is the same as if we rotated 120 degrees and followed that with another rotation of 240 degrees?

*(figure available in print form)*

We end up with the original configuration. So we need the "do nothing" operation, technically called the identity operation. If two operations result in sending us back to the starting configuration then they are called inverse operations of each other.

So a group is a set of elements (symmetries in our example), including the do nothing operation, every operation has its inverse in the set, the operations are associative and the composition of any two operations is another operation in the set.

These concepts would be used to write a program to solve Rubik's Cube. You would need to recognize the configuration and then perform its inverse operation. Memory size limitations would force you to find a way to treat many configurations as one of a type. Then you would work from your starting type to a less complicated type until you reached the identity.

One very good source on this material, and more, is the article "The Enormous Theorem", by Daniel Gorenstein in the December 1985 issue of *Scientific American* . The article is a very clear explanation of group theory, it shows the whole picture beginning from basics. There are many illustrations including the 12 symmetries of the tetrahedron with the axes of symmetry shown. It certainly gets very far along by the time it is done. So the article is a benchmark that one can use to see how much one has learned since the last time one read it and how much more one can learn as one reads it again.

So an equilateral triangle has six symmetries. How many symmetries does a square have? . . . a pentagon? . . . a tetrahedron? . . . a cube?

When asked how many symmetries a figure has we can count either configurations or axes of symmetry. When we flipped the triangle along the angle bisector that line is an axis of symmetry. If we placed a mirror along that line we would see half the figure on the paper and half the figure in the mirror, but it would look like the one original figure. Such an axis is known as an axis of reflection or an axis of bilateral symmetry. When we rotated the triangle 120 or 240 degrees it was about a line through the center of the triangle, perpendicular to the plane of the triangle, such an axis is an axis of rotation and in the equilateral triangle it is an axis of three fold symmetry.

Consider the case of the cube. It has six faces so we could put any one of the six on top and then any one of the four side faces in front. So by the fundamental counting principle there are twenty four configurations (six times four). Now see if we can get twenty four by counting the axes of symmetry. There are three axes of four fold symmetry, namely the lines through the centers of opposite faces. How many configurations does that make? Not twelve!

Remember we start with the identity so after four turns of 90° we are back to the identity. So each of these axes gives three new configurations for nine configurations plus the identity. The diagonals are axes of three fold symmetry so four diagonals give eight new positions. We are up to eighteen. Can you find six more? Take the midpoints of each of the edges. The lines connecting midpoints of edges that are parallel but not on the same face are axes of two fold symmetry. How many are there? There are twelve edges but it takes two to make a diagonal so there are six such axes. Since they are axes of two fold symmetry each one only adds one new configuration. So we found all the axes of symmetry, our answers check. (Answer 5)

When we learn that 3+5=5+3 is an example of the commutative law of addition we are apt to say, “So what?”, as if everything is commutative. Some teachers and books will talk about putting on socks and shoes. Who would put socks over shoes? Here is a “real” example.

Take a die (singular of dice). Consider these two operations: first operation, switch the left and right faces leaving the top face on top and second operation leave the left and right faces fixed but bring the top face to the front. So do the two operations in the two possible orders. Take your die with the one on top and the two on the left and the five on the right as the starting configuration. Do the operations in the order listed two goes to the right, the five to the left, and the one to the front.

*(figure available in print form)*

Put the die back in the original configuration, one on top, two on the left, five on the right. Now do the two operations in backwards order. Move the top down to the front and then switch the left and the right. The one comes to the front first and then when the left and right are interchanged the one goes to the back. So we have two different configurations, not someone improperly dressed.

As I said constructions is a year long project. Some students have seen complex numbers in Algebra before coming to Geometry. This activity will tie together the measurement of an angle by copying it repeatedly and the idea that constructions are the solutions of algebraic equations. Computers have the trig functions in their software and many hand held calculators also have them, so the students are aware of trig functions. Trig functions are one of the “out of order” topics I require my students to know about, especially the computer programming students.

To do this activity you need to know about i, the square root of -1, and the sin and cos of 30° and 60° .

Draw a set of x and y axes. Call the y-axis the imaginary axis or the i-axis. The point (1,0) will stand for the real number 1, the point (0,1) will stand for the pure imaginary number i, (-1,0) will stand for -1, and (0,-1) will stand for -i. Plot these four points. This is called an Argand diagram. Draw line segments from the first to the second, the second to the third, the third to the fourth and the fourth back to the first. What geometric figure do you see?

What angle does the line segment from the origin to (1,0) or i make with the positive x-axis? What does 90° + 90° make? Look at $i^2 = -1$. What is the angle formed by the line segment from the origin to (-1,0) and the positive x-axis? Look at -1 times i = -i. 180° + 90° = 270. Finally look at -i times i = $-i^2 = 1$. What has happened? By multiplying i by itself we have gone around in a circle by jumps of 90°. What are the solutions of the equation $x^4$ Ð 1=0 ? Does this work for other cases. Lets look at the 30° case.

My typewriter does not have a radical symbol on it so I will type 3^0.5 for the expression read as radical 3. You write the expression as you work, do not use the numerical approximation you could get from your calculator.

Plot the point (3^0.5/2,1/2) and draw a line segment from it to the origin. What is the angle made by the line segment and the positive x-axis?

What is its tan? Tan(angle)= 1/3^0.5, so the angle is 30°. The point corresponds to the complex number (3^0.5/2) +(1/2)i, lets call it z, it is too much work to keep typing it. In the 90° case we had $i^2$=-1, 90+90=180. If we calculate $z^2$ will we get 60? We get (1/2)+(i3^0.5/2) which is z with its real and imaginary parts interchanged. So $z^2$ does make 60°.

Since 30 goes into 360 12 times, z^12 should equal 1. See if it does. You need not multiply z by itself twelve times. Try z^2, then (z^2)^2 = z^4, then (z^4)^2=z^8, and finally z^8(z^4)= z^12 =1. There are two other ways to get z^12 which are easier arithmetic even though they take the same number of steps. Can you find them? (Answer 6)

So we may have a computer decide how big an angle is just as we did when we copied the angle over and over until it equaled 360. All we need to do is to translate the angle into a complex number and have the computer raise the complex number to powers until it becomes 1. The power divided into 360 gives the size of the angle.

I did not tell the whole story. If we plot a complex number and draw a line segment to the origin we will get an angle. Plot (3^0.5,1) it corresponds to the complex number 3^0.5+i. Plot it, the angle is 30°. Square (3^0.5+i), you get 2+2(3^0.5)i, plot it and determine the angle. It will be 60 degrees. However, if you raise 3^0.5+i to the twelfth power you will not get 1, you will get 4096. Why, what makes it work? (Answer 7)

The application of complex numbers to the construction of regular polygons was the work of Carl Friedrich Gauss (1777-1855). He also established which constructions would be possible and which would be impossible. He proved that the only regular polygons with a prime number of sides that can be constructed by compass and straightedge alone must be a prime number in the form 2^t+1, where t = 2^n for some n. Such primes are known as Fermat primes. Fermat thought all numbers of this form were prime. Euler showed that 2^32+1 is composite. 2^4+1 = 17 a Fermat prime. On March 30,1796, one month before his nineteenth birthday, Gauss constructed a regular seventeen sided polygon by compass and straightedge alone.

If we are engineers and we need a polygon of thirteen sides we would use a protractor to divide the 360° about the center of a circle into 13 congruent angles, extend the sides of the angles as radii of the circle . The points determined by the radii would be the vertices of the 13-gon.

The Greek geometers knew how to construct an equilateral triangle, a square, a regular pentagon, and a regular hexagon. Working from those it is possible to construct other polygons that have a composite number of sides. They did not have an exact Euclidean construction for a heptagon. They looked for one, but they were doomed never to find such a construction, seven is not a Fermat prime. Here is an approximate construction due to Heron.

Construct a circle with a chord equal in length to its radius. Construct the perpendicular bisector of the chord. Use the length from the center of the circle to the midpoint of the chord as the side of your polygon. My experience has been that I have seen hexagon constructions fail by more than the approximation construction for the heptagon.

*(figure available in print form)*

Can you prove that the approximation is in fact an approximation?

Let us combine the engineering technique with some trig. We know that angle TOP should equal 360°/14 on a regular heptagon. Compare the sin of angle TOP on the construction with the sin of 360°/14. TO=r the radius of the circle, Tp=(3^0.5/4)r so the sin of angle TOp is 3^0.5/4 = 0.433012701 while the sin(360°/14)=.433883739. The values are close, but not exact, so the construction is not exact either. I do not know how the Greeks knew this construction was not exact, but they never claimed to have an exact construction of the regular heptagon.

Here is a puzzle to conclude with. Make two copies of the network shown below. Cut along the perimeter and fold up along the lines of the square. The points labeled A match as do the points labeled B. AB will be a ridge line. Once you have your two solids the puzzle is to place one upon the other to form a tetrahedron.

*(figure available in print form)*
If you need a solution contact the Yale New Haven Teachers Institute at 432-1080 and leave a message for Jim Langan.

*(Answer 1)*

A regular polyhedron is a polyhedron whose faces are all the same regular polygon and whose vertices are congruent.

Congruent vertices have the same number of faces and the same types of faces coming together at a point. Congruent vertices can be fitted one into the other.

*(Answer 2)*

*(figure available in print form)*
The icosahedron net

*(Answer 3)*

The pentagonal star drawn with just diagonals from one pentagon. Draw the diagonals of pentagon AEIMQ to determine pentagon CGKOS. Draw the diagonals of pentCGKOS to intersect the sides of the large pentagon. Cut on ABCDEFGHIJKLMNOPQRSTA fold on SC, CG, GK, KO, and OS.

*(figure available in print form)*

*(Answer 4)*

The Archimedian Solids

Copies of the covers of The Mathematics Teacher are on file at the Yale-New Haven Teachers Institute along with networks for the construction of the solids. The colors do not reproduce well and the networks are small. It is best, to see the originals. They are available in many libraries. It will also be necessary to make larger networks to achieve solids of any reasonable size. See the literature. *Polyhedra Primer* , by Peter Pearce and Susan Pearce is very clear and informative.

*(Answer 5)*

The Axes of symmetry of a cube

*(figure available in print form)*

*(Answer 6)*

((z(z^2))^2)^2 or ((z^2)(z^2)^2)^2

*(Answer 7)*

When the complex numbers for Argand diagrams are multiplied the angles add and the hypotenuses multiply. The angle is known as the argument or the amplitude of the complex number. The hypotenuse is known as the modulus or absolute value of the complex number. When complex numbers are multiplied we add their arguments and multiply their modulii. The trick I did not tell, was to make the hypotenuse one.

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