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The Continuity Equation, the Reynolds Number, the Froude Number

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I. Physics

A. The Continuity Equation

One reason to participate in the Aerodynamics Seminar was to find examples using high school mathematics that I could share with my students. The continuity equation is such an example. When trying to solve a problem we are told to look for things that stay the same. They then can be set equal to each other giving an equation to solve. Rules like that are easy to state, but examples are easier to understand.

If we have water flowing through a pipe filled with water, the water will enter at a certain rate and leave at the same rate. The quantity pushed in per unit time will push out an equal amount. That seems like an obvious and reasonable observation. What are we taking for granted in the argument? We are assuming that water is incompressible, and its flow is steady it does not speed up or slow down during the discussion. This reasoning is using the concept of conservation of mass, namely in nature mass cannot be gained or lost in a system. A simple, characteristic example of scientific reasoning. So what can we do with it? Lets write a formula.

How would you express the rate of flow? If it were a bilge pump you would say so many gallons per hour. What do gallons measure? Volume. Volume is expressed as cubic units, cubic feet, cubic centimeters and so forth. Time can easily be changed from hours to minutes or seconds. So the rate of flow could be expressed as cubic centimeters per second. Cubic cenimeters per second could also be expressed as square centimeters times centimeters per second. What do square centimeters and centimeters per second each measure? Area and velocity, respectively. So the rate of flow could be expressed as volume per time equals area times velocity. The velocity would be an average velocity since we are using the total volume per unit time. The water in the center of the pipe goes faster than the water at the pipe wall. Can you tell what area and what velocity to use? The pipes have cross sectional areas and the water has an average velocity. So we could use the area of the entry or the exit opening. Does the argument make sense if we start with the area of the pipe and the average velocity of the water? If we multiply the cross sectional area of the pipe (cm^2) times the velocity of the water (cm / sec.) what do we get? Cubic centimeters per second, volume per unit time, a rate of flow. It makes sense both ways. Does it matter what cross sections we use? If our principle that the quantity of matter flowing in is the same as the amount flowing out then it must also be true everywhere in the pipe. Let us write this as an equation

$AV = k$ where A is the cross sectional area at a point in the pipe, V is the average velocity of the water at the same point and k is a constant, the rate of flow of the pipe, in our units cm/sec. Since the equation is true for any two points in the pipe we could also write $A1V1 = A2V2$ where A1 and V1 are the area and velocity at one point in the pipe, and A2 and V2 are the area and velocity at some other second point in the pipe. This equation is called the continuity equation. The way I visualize the continuity equation is to think of a paper wrapper of pennies. Instead of closing it, fill it full to the ends so no more will fit in. Now try pushing three more pennies in at one end. What happens? Three pennies are pushed out the other end.

We have just shown that the velocity of water at some place in a full pipe is inversely proportional to the cross sectional area of the pipe at that point. That means when the pipe narrows down the velocity of the water goes up. Something you probably noticed when you put your thumb over your garden hose.

When fundamental principles are stated one should consider them for some time. Ask what the ramifications of the principle are, what else can be proven by it? Think on it long enough so you too can see that it is “self-evident". After all, how often do you go around repeating the self-evident? If it is self-evident your listeners can figure it out for themselves. Let me share an experience where I was tripped up by this self-evident principle.

In the article "Kitchen-Sink Aerodynamics" in the book *The Amateur Scientist* by C. L. Stong the construction and use of a Hele-Shaw apparatus is explained. The apparatus may be described as a flow table. A glass plate over which a thin film of water is allowed to flow. Crystals of potasium permanganate are evenly spaced at one end to disolve setting up parallel flow lines over the table. The model, an airfoil cross section for example, is placed in the flow. We are told to put a large sheet of graph paper under the glass to make measuring the distances between the streamlines easier. We are told that the velocity of the flow is inversely proportional to the distance between the lines. I asked myself how could that be claimed? The continuity equation tells us so. The streamlines will not cross each other so when the distance between the model and the edge narrows down the flow, the streamlines will try to maintain even spacing. The distance between the lines is another way to measure the cross sectional area of the "pipe".

In scientific reasoning we should always be on guard not to apply an argument for one case to a different situation. We talked about pipes, Hele-Shaw is open to the air, it is a chanel. There are different rules for flows in pipes and flows in open chanels. The article said that thin films of water gave the same results as water flowing between two closely spaced plates.

B. Dynamic Similarity

In High School Geometry we study the concept of similarity. If two geometric figures are of the same type with corresponding angles congruent and corresponding sides proportional then the figures are said to be similar. In physics if the points in corresponding positions at corresponding times have proportional velocities and proportional accelerations then the systems are said to be dynamically similar. What could the corresponding points be? Models of ships or of canals or of river systems and their corresponding full-size prototypes. We could do model testing.

Corresponding time should be explained. If the problem being modeled is cyclical in nature, such as the revolutions of propellers, the time for the model will be the fraction (number of prototype rpm over the number of model rpm) of the real time. If there are no cyclical features, find the times for the systems to trace out similar curves. The ratio of those times is the scale factor for time. If we can model time it is possible to tell how long a machine will last. If we run its model at 20 model cycles to one proptype cycle, the model will wear out in one twentieth the time the prototype will take to wear out.

Model testing is useful, it may save lives. Lack of model testing or ignorance of model results has led to major lose of life and property. Although there was no loss of life, one famous example is the Tacoma Narrows Bridge. It twisted itself apart, the wind pushed it one way, and each time the bridge reacted the wind continued to push it in the direction of reaction, making the effect bigger. This is known as resonance, the amplification of the effect. Mario Salvadori in his book *Why Buildings Stand Up* reports that on May 17, 1854, the Wheeling Bridge over the Ohio River collapsed in a wind storm in the same way as the Tacoma Narrows Bridge did on November 7, 1840. John Roebling who designed the Brooklyn Bridge knew of the failure and designed his bridges with diagonal stays so as to prevent the twist. Figure 10 in Paul A. Hanle's *Bringing Aerodynamics to America* is a photograph of a model of the Tacoma Narrows Bridge in a wind tunnel at the University of Washington, however, the date is 1841, too late. The picture does show the wind induced hump. One reason to go to school is to learn from the experiences of those who came before us. Some models may be full size examples that are improved upon in the next version of the design.

How do we keep models dynamically similar? By running the model at the same value of the similarity constant as the prototype. What does that mean? What are similarity constants? There are many similarity constants. Similarity constants are pure numbers, they are dimensionless. Two very important ones are the Reynolds number and the Froude number. So if you want your model to be representative run it at the same Reynolds number or the same Froude number as the prototype.

There are various kinds of forces. There is the force of gravity which attracts things to the center of the earth. There are inertial forces. Inertia is the property of bodies in motion to stay in motion until opperated upon by an opposing force. One of the opposing forces is viscosity. When a bead is dropped into a jar of honey it does not fall as fast as it would in a jar of water. We say the honey is more viscous than water. We could form ratios of these forces in various ways. Two of the ratios are the ratio of inertial force to viscous force which is called the Reynolds number and the ratio of inertial force to gravitational force which is called the Froude number.

Dimensionless variables are a significant topic. Let us first learn what are units and what are dimensions. When we say a board is six feet long, the unit is feet and the dimension is length. So the dimension length can be measured in many units, feet, inches, miles, meters, *etc* . There are three basic dimensions: length, mass and time, symbolized as [L], [M] and [T] repectively. These three are combined algebraically to make the dimensions of other physical variables. For example, velocity has units such as feet per second so its dimension is [L/T], feet are units for length, per means division and seconds are units for time. Acceleration is how fast the speed changes, for example if the speed increased 3 feet per second every second we could say the acceleration was 3 ft. per sec. per sec. or 3 ft / sec ^{2} whose dimension would be [L/T ^{2}]. In Newtonian physics force is mass times acceleration so the dimension for force would be [ML/T ^{2}]. So the dimensions are multiplied and divided as in Algebra I. When a new physical variable is presented the dimension may also be stated along with it. See if you can determine the dimensions for flow and density as mentioned in the continuity equation. Can you tell what is the dimension for area?

Before defining the Reynolds and Froude numbers we need some notation. The letter g stands for the acceleration due to gravity [L/T ^{2}]. It is a constant. That is the point of the story about Galileo and the leaning tower. The Greek letter r (rho) stands for the density of the material under discussion. It is the mass of a unit volume of the material [M/L ^{3}]. Density is a property we use with bouyancy to determine the water line of a ship. The Greek letter µ (mu) stands for the dynamic viscosity of the medium [M/(LT)]. The dynamic viscosity of a medium divided by its density is called its kinematic viscosity symbolized by the Greek letter u (nu)[L ^{2} /T]. Did you get the dimension for kinematic viscosity?

C. The Reynolds Number

The ratio known as the Reynolds number, Re is

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where V is the velocity, L is a representative length such as the diameter of a pipe, the length of a ship, the chord of an airfoil, and Ã is the kinematic viscosity. What is the dimension of the Reynolds number? Substitute the dimensions for the defining variables of the number and reduce.

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The result is that the fraction reduces to one. The Reynolds number is said to be dimensionless, as would any other variable whose dimensions reduce to one. Let us use the Reynolds number in an example. We want to test a 1/3 scale model of an automobile in a wind tunnel. How fast should the wind blow? The physical relationships are the same whether the car moves or the wind moves, it is easier to move the wind. Let the subscript m mean “of the model” and the subscript p mean “of the prototype”. If the Reynolds numbers of the model and the prototype are to be equal then

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Claiming the kinematic viscosity of the air is the same in the tunnel as elsewhere we may cancel it. Substituting Lm = Lp / 3 we get

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Solvling for Vm we get

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So the model velocity must be three times the full size velocity. To model 55 mph the wind tunnel will be run at 165 mph. This is a reason to use large models. The velocity is inversely proportional to the size so if the model is smaller the wind will have to be faster, perhaps too fast. If the model was small enough we could be required to run the tunnel as supersonic speed. Even if we could (there are such tunnels) we would then have to worry about effects on the model that do not occur at subsonic speeds, such as the air being compressible and thermodynamic effects. The similarity constant for supersonic speeds is the Mach number. When there are forces that have significant effect on the model, but negligible effect on the prototype we have a scale effect. Working with larger scale models helps avoid scale effects.

D. The Froude Number

The Reynolds number is not the only similarity constant. After all viscosity is not the only force on a system. When the inertial and gravitational forces predominate then the similarity constant of concern is the Froude number. This number is also named after its discoverer. It is the ratio of the inertial forces to the gravitational forces. William Froude, a nineteenth century English scientist was one of the first to use a towing tank. Pictures of his apparatus can be found in Rouse and Ince, *History of Hydaulics* , see the bibliography. He wanted to discover the relationship between the forces on a model and the full size ship. By towing variously sized models and comparing them to each other, he found that there was no proportional relationship between the resistance of similar models and their sizes. He decided that the resistance was the sum of a frictional force and a residual force. His technique was to determine the frictional force, subtract it from the total force and get the residual force which he said was the wave making force. He determined the frictional resistance by towing boards of various lengths and finishes under water without making waves. He assumed that the frictional resistance (Rf) was proportional to the wetted surface area of the ship (S) and a power of the velocity (V) of the ship.

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His experiments with the boards where to determine the proportionality constant (f), which he called the form factor, and the exponent (n) for velocity. While experimenting with the scale models he noticed that the wave patterns were the same in number along the hull when their ratios of speed to square root of length were the same. This leads to the dimensionless constant

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where V is the velocity of the boat, g is the acceleration due to gravity and L is the length of the boat. Of course, Fn is called the Froude number. Froude did his work before Osborne Reynolds, he did not know about the Reynolds number. Let us use dimensional analysis to show that the Froude number is dimensionless. Substitute the dimensions of the defining variables (look back) and simplify to get

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So the dimensions reduce to one and the Froude number is dimensionless. So Froude’s program was to determine the frictional resistances from the wetted surface areas of the model and the ship, determine the residual resistance of the model by towing the model, scale the model’s residual resistance up to the full size ship and add it to the ship’s frictional resistance to get the ship’s total resistance. The scaling up is were Froude’s number comes into the discussion. The residual resistances varied as the displacements when the

Froude number for both the model and the prototype were equal.

If the theory makes sense it can be tested by using a finished ship. That is what was done. The Admiralty made the *H.M.S. Greyhound* available, Froude took the measurements for a model and for the full size ship, the results matched.

If viscous and inertial forces are to be similar the Reynolds number of the model and the prototype must be equal. If the inertial forces and the gravitational forces are to be similar then the Froude number of the model and the prototype must be the same. Is it possible to have both numbers equal at the same time?

Let the subscript m mean "of the model" and the subscript p mean "of the prototype". If the Reynolds numbers are equal then

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If the model and the ship are both in the same kind of water (salt or fresh) the viscosity terms divide out and we are left with

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and

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If the Froude numbers are equal

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Since g is a constant it may be canceled and

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Setting the two equations for V equal we get

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Which says the only way the Reynolds and the Froude numbers can be equal is if the length of the model and the length of the ship are equal, that is if the model is full sized. Does this mean that model testing is impossible? No, just that there are limitations to the modeling process, they are approximations in some aspects. The engineer needs to know what is being modeled and make corrections for the effects that are not being modeled. Whether the Reynolds or the Froude numbers are kept equal depends on which effects are considered significant. Waves are caused by gravity, so if waves are involved the Froude numbers are kept equal. If we are submerged in a medium such as air for airplanes and cars or water for submarines and propellers the Reynolds numbers are kept equal provided the air speeds are moderate.

II. History

At the Sound School we try to integrate maritime topics into our curriculum, to use Long Island Sound as a theme for our teaching. Hoping that students will be motivated to greater study of a topic if they see a use for it. Here is some history of naval architecture related to aerodynamics. Aerodynamics and hydrodynamics are both part of one field called fluid mechanics. Many of the principles of fluid mechanics were known before airplanes existed. These principles had been discovered in hydrodynamics. One apparatus of investigation was the towing tank.

A. Towing Tanks

Much of the hydrodynamics was motivated by the building and maintenance of canals. In fact Benjamin Franklin built a towing tank to test the observation he had made on the canals in Holland that boats go slower in shallower water. The tank was in fact a narrow wooden trough, the model was towed by a weight falling over a pulley at one end. Even though Ben Franklin was a founder of the United States, the U. S. Navy did not get Congress to approve funding for a tank until 1896. The tank was built at the Washington Navy Yard under the supervision of Naval Constructor David Watson Taylor who direced it for the next fifteen years. Much significant work was done.

The claims of the advocates of model testing were substantiated early on. In 1902 the Model Basin designed two armored cruisers of 14,500 ton displacement that were 820 tons heavier than similar predecessors but were able to cruise at 22 knots with less horsepower while consuming less fuel.

Taylor instituted the practice of using wooden models instead of wax models as used by other naval architects. This gave more accurate measurements, and avoided models melting in Washington, D. C. summers. It was much more expensive, however, $80 against fifty cents for wax that could be melted down and used again. He was responsible for the bulbous bow to dampen the bow wave thus decreasing wave resistance. This type of bow was first used on the USS *Delaware* in 1907 with great success.

B. The *Great Eastern*

The technological "giant steps" of any historical period determine the topics of scientific research for that period. The *Great Eastern* was one such giant step. Construction of the *Great Eastern* started in 1854, her launching began in November, 1857, and she finally floated at the end of January, 1858. She was 680 feet long the next longest ship of her day was 380 feet. Her length was not exceeded until the *Oceanic* in 1899 and her tonage was not exceeded until the *Lusitania* in 1906. The reason it took three months to launch her was that metal rails where used for the ways and the cradles were iron shod. So much heat was generated by the friction of iron against iron that the cradle shoes and the rails welded themselves together in November at the first attempt. The ship was jacked down the ways an inch at a time after jacks were designed and built.

She was the only vessel ever built that had sails, paddle wheels and propellers, with the paddle wheels and propellers having their own independent engines. She burned a ton of coal per mile. She had a capacity of 12,000 tons of coal. She was under powered, she had about 2600 horsepower with a top speed of 14.5 knots on a displacement of 27,000 tons. Remember, no one had ever done this before, there were bound to be mistakes, and unforseen problems. All her problems pointed out the need for even more scientific investigations. Her builders had been successful in their previous ventures building other ships, rail roads and bridges.

One engineer associated with the *Great Eastern* was William Froude. From his experiences grew his life's work the study of the powering of ships. Other names to research are J. Scott Russel who designed her and built her in his ship yard. The name that most people associate with the *Great Eastern* is Isambard Kingdom Brunel the owners, technical advisor and probably the top engineer of the time. It is a question of historical research as to how much Brunel contributed to the design, it is called "Brunel's great ship" and its misfortunes are said to have killed him.

One succes of the *Great Eastern* was the laying of the transatlantic telegraph cable after the Civil War. To learn more see the article by Chiles in the Fall l987 issue of *Invention & Technology* .

C. *Turbinia* and Cavitation

In search of more speed more efficient engines were built and placed in ships. The first turbine powered ship was Sir Charles Parsons' *Turbinia* built in l894. Parsons had built the first successful turbine to power a dynamo, an electric generator. He had done model tests and had great expectations for the ship, the turbine was very powerful. The results were disapointing. The higest speed recorded was less than 20 knots. The problem was cavitation, a phenomenon recognized and named by William Froude. The propellers were spinning at l8,000 rpm, as they spun they decreased the pressure until the water became vapor, a bubble, a cavity. The power was going into making the bubble instead of pushing the boat.

The remedy was to operate at lower rpm with more turbines and propellers. The original design was one turbine with one shaft of three propellers. The successful design was three turbines each having a shaft turning three propellers, which achieved the speed of 34.5 knots in 1897.

III. How to use this Unit.

As stated previously I want my students to see uses of mathematics. One way to do this is to read about uses of mathematics such as engineering projects and their solutions. I see this unit as something the student can read as a start on the subject. The unit may be broken into sections for reading or the whole unit may be given to the class. I want discussion with the students on the writing: Is it clear? How could it be improved? What was left out? Did some source over simplify the story? After disecting this unit it will be their turn. They will write a report on some technological project.

Learning is a do-it-yourself job. The required work may not be enjoyable. I hope the collateral reading will be enjoyable and fascinating.

I see the student projects centering on the history of engineering, ship building, naval architecture and technology. Books that are readily available are the Time-LIFE Books *Ships* and Thomas C. Gillmer's *Modern Ship Design,* both are public library books.

The most important part of the process will be the discussion. Students are all too willing to sit passively by as if they were jugs waiting to be filled up. I see the reading as the motivation for the discussion. If the books are interesting enough the students will be willing to share them with one another.

I will use this material when we discuss variation in Algebra II. This unit itself mentions direct and inverse proportion. The naval architecture readings give examples of variation when they discuss laws of mechanical similitude, such as the wetted surface varies as the square of the length on the water line. There are other places that the unit could be used, but I need a starting place. If the students show interest the project will expand.

The Student's Bibliography

I claim my students will find these books and articles to be readable . I believe the students will also find them interesting.

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The Teacher's Bibliography

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