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Crystals: Geometry and Groups

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I. Introduction

A. Why Study Crystals?

What are crystals? The answer to that question is the end result of a great deal of research and study by many people over many years. Crystals can be gem-like solids as seen in quartz crystals. So one might believe that this leads to the study of the regular and semi-regular solids of geometry, but crystals are more than that. Crystals are systematic arrangements of molecules to form physical solids. Crystals are solid state physics. When we look at the gem-like object we are looking at a single crystal, when we look at a slab of marble or a sheet of steel we are looking at many crystals all next to each other, polycrystalline forms. Crystals are beautiful, everyone likes beauty. Why are crystals beautiful? Crystals have symmetry.

There are many topics from mathematics that relate to crystals and many topics of crystallography that are math. How can crystals be represented as three dimensional objects? There are many answers to that question: Miller indices, space groups of symmetries, drawings, projections.

Crystals are the evidence of the atomic structure of matter. There are three laws of crystal morphology:

1. Crystals grow naturally with plane faces.
2. The angles between faces of a crystal are characteristic of the crystal.
3. Law of Rational Positions. When three noncoplanar edges of a crystal are used as coordinate axes, the coordinates of the points where the planes of the crystal faces intersect the axes can be expressed in small integers.

If we postulate that crystals are made of congruent building blocks stacked one upon the other, one next to the other, each of the laws can be derived.

When we were taught the scientific method in elementary school we were told scientists collect data and then interpret it to get theories. So one would expect those laws to have come first and then the theory. Actually the history is more complicated. The scientists had other theories (like alchemy) and were looking at the data from those points of view. The data were available but the interpretation took time.

It seems that postulational systems are a way of thinking that people naturally follow. The laws or postulates are simplifying principles that explain and predict what happens. If the predictions don't match up it takes time for people to reject the old laws and look for new ones.

B. Growing Crystals.

Another reason to study crystals is to grow them. Growing crystals is an opportunity for students to have a hands on activity. The two references to use are *Crystalls and Crystal Growing* by Alan Holden and Phylis Morrison and *Crystals—A Handbook for School Teachers* by Elizabeth A. Wood.

I should have started this activity much earlier. I should have set it up in the classroom last April or May to see how my students would react to science apparatus in a math room. I have grown an alum crystal. The process of getting the seed is satisfying by itself.

To get seeds one dissolves the alum in some water and then places the solution on a shallow plate to evaporate, the shallower the plate the better. Examine your seeds with a magnifying glass or even a microscope. Philip and Phylis Morrison in their book *Ring of Truth* suggest watching the seeds form under the microscope as the water evaporates.

Once you have the seed tie it with a thread and suspend it in a supersaturated solution and wait. The more seeds we have growing the better our chances of seeing a big, beautiful crystal. So see if you can inspire your students to try growing crystals independently.

If you are not a science teacher you may not know that Macalester and Bicknell, the chemical supply company on Henry Street in New Haven will not sell chemicals to private individuals. So either see your friendly science teacher or plan far enough ahead to get a school supply purchase order. Meanwhile, grow crystals with the chemicals you have in the house or can get from the supermarket: alum, sugar, salt, borax, epsom salt or what have you.

II. Science

A. The History and Theory of Crystals

Let us look at the history of the study of crystals. If you research the history you find out that the story is not as clear cut as the short introductions make it appear. An important idea may have been mentioned by a person but he may not have given it much emphasis at the time and instead have been concerned about some other part of his theory which might even have been wrong.

The study of crystals has been the concern of mineralogists. In fact, one of the most accessible sources is Dana's *Manual of Mineralogy,* you will find it in most public libraries. James Dwight Dana (1813-1895) and his son Edward Salisbury Dana (1849-1935) were Yale professors. In 1837 the father wrote the standard mineralogic reference *A System of Mineralogy,* the son kept it up to date and it is still being kept up to date by others. It is available in your local library. The Manual is a good source of illustrations. Hopefully, this unit will be an introduction of crystallography to readers who will want to find out more.

Another major source is Martin J. Buerger the inventor of the precession method of x-ray diffraction who has written a number of books on crystallography. The biographical notes on authors in the collection *Fifty Years of X-ray Diffraction* tell us that his field of study was geology and mineralogy.

Crystallography in many respects is another example were the theory or mathematics existed long before the "reality". In 1912 with the birth of x-ray diffraction the demonstration of the theory became visible.

In 1669 Nicolaus Steno in his Latin publication *De Solido intra solidum naturaliter contento Dissertationis Prodromus* described how he took sections of quartz crystals and measured the angles. He found that the angles were the same no matter how long or short the sides. He did this by cutting the crystal perpendicular to an edge to get a wafer which he then traced onto paper. If he did this to different quartz crystals, he got the same angles when he cut wafers at corresponding edges. For this he gets credit for the law of constancy of angle also known as Stensen's Law from his name in his mother tongue. This was not the main point of his book, he did not say anything about crystals of other substances.

The law of constancy of angle was stated by Rome de I'Lsle in 1783 and says that for all crystals of the same substance the angles between corresponding faces are congruent. Can you propose a reason why nature works this way? What makes it happen?

The next story is like Newton's Apple; there is controversy that it ever happened. The story goes that one day in 1784 while examining some crystals Rene Just Hauy dropped a specimen and one of the larger crystals broke off. He noticed that it had broken to form a face. He tried, and succeeded, to cleave it in other directions. The piece that remained was not the same shape as the original crystal. He claimed the original crystal was made of building blocks shaped like the nucleus achieved by cleavage. Read the *Origins of the*

Science of Cystals by John G. Burke to find out more. If you read Burke you will find that many people worked on the subject and said many things that turned out to be true later. Do they deserve credit for stating the principle? Did they really know what they were saying?

The text books that give historical notes surely make the history sound certain. It is not so clear when you try to read the whole story. Often times the “discoverer” was arguing about something else. Crystals are ice that is too cold to melt. Crystals grow like plants. The history seems to be evidence that civilized people search for ways to explain nature by logical theories.

So Hauy wrote some articles and books and showed how large crystals could be built out of little cubes to get shapes that were not cubes. He did this by stacking the cubes at certain slopes. Put down a layer, then go in 2 cubes before starting the next layer, and so forth. Different faces call for different steps. See the mineralogy books for pictures of Hauy’s models.

From this time crystallography became an exact science. The interfacial angles of crystals were accurately measured. The device to measure the angles is called a goniometer. There are various types. The simplest goniometer, called a contact goniometer, is a semicircular protractor with an arm pivoted at the center. Place the base against one face and the arm against another and read the angle off the protractor.

(figure available in print form)
In 1809 William H. Wollaston invented the reflecting goniometer which allowed the measuring of angles on much smaller crystals, even ones with rough faces.

So angles became the big issue with crystallographers. The length of the edges did not matter it was the angles. The shape of the crystal as far as lengths are concerned is called the habit of the crystal. Crystallographers visualized the crystal at the center of a sphere with lines radiating out perpendicular to its faces to intersect the sphere at points which could be mapped like longitude and latitude. Then they decided to map the sphere onto a plane using the technique called stereographic projection. One place we all have seen stereographic projections is the trade mark of the National Geographic Society. The portion of an astrolabe that does not move is also a stereographic projection. The stereographic projection is a topic of study unto itself.

If crystals are made of bricks all stacked next to each other what can we say? We can explain the three laws of crystallography. The bricks will be so small that the steps will not be visible to the naked eye thus giving flat faces. The stacking of the bricks will give fixed slopes which in turn give fixed angles. Notice that a stacking schedule of one over and two up will have a tangent twice that of a stacking schedule one over and one up. The tangents will double but not the angles. The tangents will be integral multiples of a fundamental value, but not the angles. The edges of the bricks will line up to give a three dimensional coordinate system and the dimensions of the brick will give the units to use along each axis.

What shape will the bricks have? If they were cubes then our coordinate system would have three axes at 90° to each other with equal unit lengths along each axes, just like analytic geometry. Crystals are not so simple. There are crystals with cubical building blocks, they form the cubic or isometric system of crystals. Other building blocks form other systems of crystals. Can you determine the possible shapes of the other building blocks? There are only five more.

We could have traditional rectangular bricks where the lengths of the edges are three distinct values. Such bricks make the orthorhombic system. We could tip our traditional brick at an angle so one dihedral angle was not 90° this makes the monoclinic system. We could tip it in three directions to make the triclinic system. We could take the cube and change the length of one dimension to make the tetragonal system. We could use bricks that did not have all rectangular faces, we could use hexagonal tiles to make the hexagonal system. That makes the six crystal systems.

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(figure available in print form)

The reader might wonder why triangular prisms do not make a crystal system. We said that crystals are bricks stacked next to each other, we meant touching along faces without turning the brick. This idea is called translation. If we were to place triangles along a line there would be empty “upside down” triangular spaces between them. To fill those places we would have to turn the triangle as we moved it. So we put two triangles together to get a parallelogram and the monoclinic or triclinic systems.

Knowing the system of a crystal scientists can predict how physical properties will behave. For example, a cubic crystal will expand equally in all direction when heated, while a tetragonal crystal will expand a different amount in one direction.

Let us think some more about the bricks. If you were at any corner of any brick you would not be able to tell it from any other brick. The bricks would have to be of proper shape. You might be able to define directions, some points could be closer to you than others, but then again maybe not. You might be able to look at the system in certain ways and still see the same arrangement as before, stand on your head, turn and look back, maybe do both. These tricks lead to the concept of transformations: translations, reflections, inversions, rotations, which in turn lead to the mathematical structure called a group. A translation is when you slide from one brick to the next ending up in the same configuration you started at. If you continue in that direction each time you go the same distance you will end up in the same configuration. Reflections are like mirrors. Two points are reflections of each other if the mirror is the perpendicular bisector of the line segment connecting them.

Inversion is the hardest to describe. You need a center of inversion, a point. To find the inversion image of a point you draw a line segment from the point to the center of inversion, then you continue the line on to a point as far from the center as the original point. It is as if the mirror had been shrunk to a point. The center of inversion is the midpoint of the line segment joining the original point and its image.

A rotation is the easiest to describe. You pick a line to serve as if it were the axis of a top and rotate the configuration about it. All the transformations can be described in terms of the coordinates of the points. You start with one point and you get another point called the image of the first.

All these transformations may be expressed in terms of what they do to the coordinates of any point. My experience was that the coordinate explanation for inversion was much easier to understand than the verbal: point (x,y,z) becomes (-x,-y,-z).

So transformations lead to groups, so let us tell _____

III. Mathematics

A. What is a Group.

If we placed an equilateral triangle on a piece of paper and made a tracing of it, we could pick it up and put it back down again and not know if we had changed its orientation. What we would have done is called a rigid motion or an isometry. To be able to tell what happened we could label the corners of the cutout and the corners of the tracing.

One question to ask is "How many ways can we pick up the cutout and put it down?" We could rotate it around its center to move a corner one space in a counterclockwise direction, or two spaces, or even three to come back to the starting position. Some obvious things to note are that we can do one thing and follow it with another and it is as if we had done a third thing in the first place. Two moves of one space each are the same as one move of two spaces. There are a limited number of moves. If we keep moving we end up with previous positions. One move is a waste, if we rotate three spaces it is as if we had not moved at all.

When we have situations like this in math: where things combine to make new things of the same kind we say the system is closed. So our system of three rotations is closed. The move that is the same as doing nothing is called the identity operation. The fact that every operation can be undone to get back to the original position is described by mathematicians by saying each operation has an inverse. When we have a closed system with an identity element and each element has an inverse, we say we have a group. Technically a mathematician would want another additional property: the operations should be associative, a(bc) = (ab)c. Isometries are associative. It took some historical time for mathematicians to recognize that there are non-associative "things". So we will be historical and not check for associativity. Division is an example of a non-associative operation. Example: 12 Ö (6 Ö 2) makes four, while (12 Ö 6) Ö 2 makes one. Check and see.

(figure available in print form)

Do the three rotations make the only group for our equilateral triangle? Are there more isometries for the equilateral triangle? Can we move it without rotating it? Flip it along an altitude so the two base angles switch positions but the vertex angle stays fixed. You will have to put labels on the back of your cutout to match the ones on the front. How many flips are there? Do the flips make a group? We have three vertices so we have three flips. Some might say we have four flips, the identity flip of making no flip at all. So, do the four flips make a group? What happens if we flip around the top vertex and then flip around the lower left hand vertex? Do we get a flip? No, we get a rotation, it is as if we rotated two spaces counterclockwise. So the flips are not closed, they do not form a group by themselves. Along with the rotations and the identity the flips will make a group. To verify it make a multiplication table. Let I stand for the identity, R1 stand for a rotation of one space counterclockwise, R2 for a rotation of two spaces counterclockwise, FI for a flip around the top vertex, F2 a flip around the lower left vertex, and F3 for a flip around the lower right vertex.

(figure available in print form)

To read the table find the first operation in the first column and then read over to the column with the second operation on top, there is your answer. For example, operation R2 followed by operation FI is operation F3. Notice that operation FI followed by R2 is operation F2. Our operations are not commutative. The table helps us see that the system is a group. It is closed because no answer is a new thing, all the answers are operations listed on the edges. There is an identity one column matches the left hand column and one row matches the top row. Every element has an inverse because the identity appears once in each row and column and even when the identity is not on the main diagonal the two operations still commute (R1 followed by R2 = R2 followed by R1 = I).

So we now have a bigger group the identity, the rotations, and the flips. The bigger group contains the group of rotations, so the group of rotations is called a subgroup of the larger group. Are there any other subgroups in the bigger group? Since each flip is its own inverse we could use a flip and the identity to form a group. We also could look at the identity alone and consider it a group. So a group is a set of things that operate on each other always getting themselves for answers, (just saying closure again).

The process of lifting up the figure and putting it down again, an isometry, is also known as a symmetry operation. The symmetries that required us to use both sides of the triangle, the flips, are called improper symmetries.

Here is a problem that might be more obviously relevant to us, where the symmetries tell us something. The queen in chess can move horizontally, vertically, and diagonally. If we had eight different queens could we place them on an eight by eight chess board so that no queen could capture any other queen. You do not need to know anything about chess to solve the question. The question could have been: place eight pieces of some kind on an eight by eight board so that no two pieces are on the same row, the same column, or the same diagonal.

I will not solve the 8 by 8 case for you. Look at the 4 by 4 case. Here is a solution.

(figure available in print form)

1 2 3 4

2 4 1 3

My symbol means a piece is in the first row second column, another piece is in the second row fourth column and so forth. The number on top gives the row, the number underneath gives the column.

Is this the only solution? Did you find a different one? Are there any different solutions? This is where we apply the symmetries. We can rotate the board ne quarter, one half, or three quarter turns: R1,R2, and R3, we can flip it on its horizontal midline: MH, its vertical midline: MV, or one of its diagonals: D1 and D2.

R1 gives 1 2 3 4
2 4 1 3

(figure available in print form)

R2 gives 1 2 3 4
2 4 1 3

(figure available in print form)

R3 gives 1 2 3 4
2 4 1 3

(figure available in print form)

MH gives 1 2 3 4
3 1 4 2

(figure available in print form)

MV gives 1 2 3 4
3 1 4 2

(figure available in print form)

MV gives 1 2 3 4
3 1 4 2

(figure available in print form)

D1 gives 1 2 3 4
3 1 4 2

(figure available in print form)

D2 gives 1 2 3 4
3 1 4 2

(figure available in print form)

What is going on here? Are R1, R2, and R3 the same operation? They give the same result. What about MH, MV, D1, and D2? Let us start with a different configuration, one that is not a solution, and perform the operations and see how many new configurations we get.

I is 1 2 3 4
3 1 2 4

(figure available in print form)

which is not a solution since the first column and the second column pieces are on the same diagonal.

R1 gives 1 2 3 4
4 1 3 2

(figure available in print form)

R2 gives 1 2 3 4
1 3 4 2

(figure available in print form)

R3 gives 1 2 3 4
3 2 4 1

(figure available in print form)

MH gives 1 2 3 4
4 2 1 3

(figure available in print form)

MV gives 1 2 3 4
2 4 3 1

(figure available in print form)

D1 gives 1 2 3 4
2 3 1 4

(figure available in print form)

D2 gives 1 2 3 4
1 4 2 3

(figure available in print form)

all distinct results. There were only two answers in the first case because the first arrangement was so symmetrical. The different operations really are different. Can you find a solution for the eight by eight board? I have one solution that transforms into eight distinct solutions. Is it the only one? I do not know.

Just as we asked how many bricks are possible we also could ask how many groups are possible. The answer to those questions is for another time. To get started on the answers one needs to have names for the groups. Here is one technique.

In our triangle and square examples we had axes of rotation, a three-fold axis for the triangle and a four-fold axis for the square. When we flipped the figures we had two-fold axes of rotation. These operations can be symbolized as 1,2,3,4,5,6, . . . n, for n-fold axes of rotation. 1 means you rotate right back to the starting position. In two dimensions a two-fold rotation has the same result as a reflection in a mirror, which is not the case in three dimensions. The symbol for a mirror is m. This gives names to our two groups: 3m for the triangle and 4mm for the square. The number tells the type of rotation axis and the m indicates a mirror. One might wonder why there is only one m in 3m, while there are two m's in 4mm. After all, there are three mirrors on the triangle and four mirrors on the square. Either 3m is correct and 4mm has an extra m, or they are both two m's short. The answer is due to the way the mirrors interact with the rotations and each other. It is logical, ask me to show you with a diagram. The symbols for the groups are known as the Hermann-Mauguin symbols.

I would have liked to have spent more time on Hermann-Mauguin, however I only got it straight in my own head as this project was coming to an end. I even found some useful references in books I had looked at earlier. So I must have learned something.

One special reference for groups and Hermann-Mauguin notation is *Symmetry* by Ivan Bernal, Walter C. Hamilton, and John S. Ricci. The book comes with a stereo viewer, in a pocket on the inside back cover, so one can actually see the three-dimensional crystallographic point groups in three dimensions. Read the book as well, and read it Slowly, with thought.

B. Miller Indices

Crystallographers need ways to describe crystals. One way is to draw pictures, but pictures are hard to copy and write down every time you want to talk about a crystal. Another way is to have numbers associated with the crystal. In fact the numbers will often be found on the pictures so you will know more certainly the orientation of a particular face.

A standard way to attach numbers to figures is to set up a coordinate system. So we are back in Algebra I and Algebra II with graphing. We need a third axis for three dimensional solid objects so we have the z-axis coming out of the page. In Algebra I we learn that the equation of a line whose x-intercept is a and whose y-intercept is b is *x* + Y =1. a b In three dimensions the equation

x + y + r = 1 a b c would be a plane that cuts the x-axis at a, the y-axis at b, and the z-axis at c. The three numbers, (a,b, and c) could have been used as indices for the plane. Furthermore, since the crystallographers were only interested in the angles the three numbers could be “reduced” if they had a common factor, giving a parallel plane. Also, since no one likes fractions, the equation was multiplied through by abc to give bcx + acy + abz = abc. The numbers bc,ac, and ab are called the Miller indices h,k,1. These indices will be integers and usually single digits. They are written without commas separating them except for the rare occasion when one is more than a single digit. When the intercept is negative the index will also be negative. To show a negative value a bar is placed over the number. Remember, Miller indices will always be whole numbers: positive, negative, or zero.

If you are reading the descriptions of a crystal you will want to translate the Miller indices into intercepts. To change from Miller indices back to intercepts take the reciprocal of each digit and then multiply each by a common denominator to make whole numbers. The numbers will be in the order of the axes, *i.e.* x-intercept, y-intercept and z-intercept.

When saying Miller indices in words, 111 is said as one-one-one, not one hundred eleven.

In Elizabeth A. Wood’s book *Crystals and Light* she has a picture of a pyrite crystal which she shows as a pentagonal dodecahedron.

(figure available in print form)

She gives the Miller indices of the faces that are visible as 102, 021, 210, 102, 021, 210 and says the point-group symmetry is m3. One is a convenient index since it is its own reciprocal. Zero is not so obvious. The reciprocal of zero is commonly called infinity. So the plane intersects the axis at infinity, or in other words it never intersects the axis, it is parallel to the axis. So we only have to change the indices + 2 and + 1 to their reciprocals + 1/2 and + 1 which become + 1 and + 2 when we multiply by the denominator to get whole numbers. Yes, they are the same values as we started with, but the order is not the same and that means different axes.

(figure available in print form)

(figure available in print form)

The group symbol, m3, translates into a cubical system with the four diagonals of a cube as axes of three-fold rotational symmetry and the x-y, x-z, and y-z planes as mirror planes. We should be able to cut a model of the crystal out of a cubic 4 by 4 by 4 block. See the plan at the bottom of the previous page.

Here is some explanation. Follow the steps. K is the midpoint of AB, L is the midpoint of FG. Similarly N,M,J, and I are midpoints of their respective segments. Next draw a line through V parallel to JI, a line through T parallel to KL and a line through 0 parallel to NM. Those lines will serve as if they were the ridges of house roofs. Next we need to find the eaves lines and the gable lines. S is the midpoint of BM and R is the midpoint of GO. The story is the same for P and Q. Finally, using the corners as axes of three-fold symmetry rotate each of the ridge lines into the position of KL and mark in the eaves and gables. Now cut off the wedges to get the gable roofs.

How do you cut it out? Make a jig to keep your fingers away from the saw. How do you keep your angles when your guide lines are cut off? I made mine out of clay, cutting it with a wire and putting the cutoffs back to maintain the cubical guide shape. After all the cuts were made then the wedges were peeled off to get the dodecahedron core. The force of the cutting distorted the clay block. I did get a pentagonal dodecahedron, but

I can not call it regular nor can I claim that it would not be regular if the process were more precise.

A drawing showing all the lines is shown in figure 8.

I would not expect anyone to follow the drawing unless they drew it too.

C. Projections: Mechanical Drawing

One of the objectives of geometry is to visualize objects in space. How can this be taught? Let the student experience drawing three dimensional objects. Mechanical Drawing is a way to draw even if one believes one is not an artist. Mechanical Drawing is math, it is part of projective geometry. See Morris Kline *Mathematics in Western Cultures* . John Pottage in *Geometrical Investigations* has a number of problems that are solved by mechanical drawing techniques, including a copy of woodcut by Albrecht Durer showing how to construct an ellipse as a conic section by mechanical drawing techniques (page 436).

Look at figure 1. In the upper right we have three pieces of overhead projector transparency film (VWDU, UDST, and WDSR) joined at their edges to form a corner, we place a block in that corner and trace onto the plastic the edges that touch the plastic. Now we unfold the plastic corner and have three views: top view, front view and side view. You have done a parallel projection of each side onto a picture plane namely its piece of plastic. This is all there is to mechanical drawing and blue print reading. So, how can mechanical drawing be a full year course? Easily, the “block” could be much more complicated, needing auxiliary views, section views, shading, maybe even the shadows cast by the “block”. Also time is needed for practice to gain speed, while achieving accuracy and neatness.

Let us look more closely at the figure. How can we improve it? We do not need to show the “pieces of plastic”. We could have some space between the three views so it would be more obvious where each view begins and ends. Notice how D,U, and C appear on both the x and y axes. If you were to draw a line segment from the U on the x-axis to the U on the y-axis you would form a triangle. What kind of a triangle? Notice the dashed lines. The dashed lines stand for invisible edges. Edges HG,FH, and HB are the back edges of the solid block that we would not see in reality.

Let us think some more about this. When we have plans what do we want to get from them? The sizes of the dimensions that make our object, the angles we have to set our saws at to get the pieces to fit. Will these dimensions be on our pieces of plastic? Think about it.

Let us explain “parallel projection” in more detail. If we keep the model in the corner we will be rather cramped, at least we will have one line right on the edge of the plastic sheet, so let us move the model out of the corner. Now we rest our pencil on an edge with its point touching a plastic sheet, slide the pencil along the edge keeping it parallel to the other plastic sheets. You now have a line on your plastic, go completely around the model and you will have one of your three views as before. Let us change our point of view. Look at the pencil; it is always perpendicular to the picture plane. So we could look at the process as keeping the pencil perpendicular to the picture plane and tracing the edges of the model with the “other end”. I say the “other end” because the pencil would have to change its length as the line got closer or farther from the picture plane. Figure 2 is my attempt at illustrating this.

Let us draw a block with some faces not parallel to the picture planes. Turn the model so it “rests” on one of its long edges. See figures 3 and 4. A rectangular brick has been placed in the interior of a plastic box and parallel projected. The subscripts tell what view the point is on, t,f, and s for top, front, and side views. Points with the same capital letter correspond. Figure 4 is the box in figure 3 unfolded. Edge HK is invisible from the front so it becomes a dashed line in the front view. Likewise edge EG is underneath so it is invisible from the top and is a dashed line in the top view. Look at edge CB on the model. Is it equally as long on each of the views? Is it full size on any of the views? If line segment CB on the front view were called x, and if line segment CB on the top view were called y, and line segment CB on the side view were called z, what would be the relationship between all three variables? Why?

So what kind of lines will not be the same length on the plans as they are on the model? Lines that are not parallel to their picture planes.

Let us look at another example. We have a block with one corner knocked off (figures 5 and 6). Knocking the corner off leaves a triangular face FDE. In the three view mechanical drawing there is no full size congruent image of the triangle. In each view only one of the diagonal lines is full size the other two are shortened. If we want to see the true shape of the triangle we have to *develop* it, one of those auxiliary views mentioned earlier. A useful reason to develop all the faces of a figure (other than clarity) is to have a pattern one can cut out of paper to form the object. The paper cutout is a check to see if your drawing is the figure you claim it to be. Figure 7 is the development of the three visible faces in figure 5. It leaves out the back, bottom and left side of the block. When it comes to developing the block, to make a model, we want the matching edges back together again. I chose to put the DC edges together so there would be more room for the triangle with ED as its base. Once the top, front, and side faces are together it is time to make the triangle. With the point of your compass at D set the radius to DF from the top view and draw an arc. Then with the point of your compass at E set the radius to FE from the front view and draw an arc to cross the previous arc. The intersection is F, the vertex of the triangle. The construction is the same as constructing a triangle given three sides, as in geometry class.

IV. How to use this Unit.

When one studies group theory in mathematics classes one is told groups are used by crystallographers. The math texts then go on to talk about other problems. Some abstract algebra books prove there are just 17 possible wallpaper groups, some even have short proofs. Short proofs are the result of building up a lot of notation and lemmas, not something to share with non-math majors. So I wanted to find out what crystallographers did with groups and perhaps find out how it is shown that there are just only so many ways molecules could be arranged to form crystals.

I wanted to find opportunities for students to read on their own, even to do independent projects. This is an introduction of some topics that lend themselves to independent study. Crystals are interesting by themselves, students can grow the crystals and will be moved to ask questions. Finding answers to those questions might motivate some students to become scientists. Mechanical drawing is a skill that one can teach oneself, a skill that can teach one problem solving, and a skill that one becomes proficient at by independent work. There are books on group theory that claim to be accessible to high school students. By working on this unit I hope I have made an introduction that will help students to work through the references. The books I consulted say a great deal more than what I was able to report here. There is more in those books than I got out of them from my first readings of them, so I intend to go through some of them again. Especially Elizabeth Wood's *Crystals and Light* .

This unit gives a starting point for my students to find projects for themselves. By starting with this material we will have some common vocabulary. My students can assume that I know the vocabulary and concepts in this material, but anything they discover that is not in here will be news to me, that is, something for them to teach me.

This unit was an opportunity to find problems and concepts to introduce or allude to when teaching the standard math curriculum. Answers to the question: "What good is this stuff?" The law of constancy of angle points out the value of trig functions over just angles. The tangent is the "obvious" variable not the size of the angles.

Matrix multiplication is taught and students need time to practice matrix multiplication before they use it in larger problems. One early practice activity would be to multiply all the 3 x 3 matrices having a one in each row and column and zero in all other positions. No two ones in the same row or column. It would be a chance for the students to see closure.

If you want a book list I have a data base of the books I looked at. It tells what libraries they may be found in. The bibliography is books I believe others can easily enjoy.

Feedback is welcomed. Leave a message at the Sound School.

The Teacher's Bibliography

Martin J. Buerger *Elementary Crystallography* John Wiley & Sons, Inc., New York, 1956.

Martin J. Buerger *Contemporary Crystallography* McGraw-Hill Book Company, New York, 1970.

If a prize is to be awarded to the author with the most number of books consulted by me, Buerger wins it.

F. C. Phillips *An Introduction to Crystallography* Longmans, London, 1963.

A book I intend to purchase. Frequently cited in the literature. Even show how to draw crystals.

The Student's Bibliography

I claim my students will find these books readable. I believe the students will also find them interesting.

Ivan Bernal, Walter C. Hamilton, and John S. Ricci *Symmetry A Stereoscopic Guide for Chemists* W. H. Freeman and Co., San Francisco, 1972.

To repeat, a great book, even if you only look at the pictures.

John G. Burke *Origins of the Science of Crystals* University of California Press, Berkeley and Los Angeles, 1966.

A history, shows how the story is more complicated than the textbook versions.

Rodney Cotterill *The Cambridge Guide to the Material World* Cambridge University Press, Cambridge, 1985.

An abundance of colorful illustrations. Anything else I could say would sound like I was quoting the publisher's blurbs.

James Dwight Dana revised by Cornelius S. Hurlbut, Jr. *Dana's Manual of Mineralogy, 17th ed.* John Wiley & Sons, Inc., New York, 1959.

Cornelius S. Hurlbut, Jr. and Cornelis Klein *Manual of Mineralogy (after James D. Dana) 19th Edition* John Wiley and Sons, New York, 1977.

Bruno Ernst *The Magic Mirror of M. C. Escher* Ballantine Books, New York, 1976.

An aspect of the unit not pursued. Escher did many prints as space filling patterns. They can be analyzed as to what group structure they have. Escher also investigated perspective which this book discusses.

Paul P. Ewald *Fifty Years of X-ray Diffraction* International Union of Crystallography, Utrecht, The Netherlands, 1962.

Alan Holden and Phylis Morrison *Crystals and Crystal Growing* The MIT Press, Cambridge, Mass., 6th printing, 1988 new material copyright 1982 by MIT, copyright 1960 by Alan Holden & Phylis Morrison.

A classic, everyone should have a copy.

Alan Holden *The Nature of Solids* Columbia University Press, New York, 1965.

Morris Kline *Mathematics in Western Culture* Oxford University Press, New York, 1953.

Robert Lawlor *Sacred Geometry* Crossroad, New York, 1982.

Many figures, especially constructions.

Josef Vincent Lombardo, Lewis O. Johnson, and W. Irwin Short *Engineering Drawing* Barnes & Noble Books, New York, 1956.

A reference students can work through on their own.

Philip and Phylis Morrison *The Ring of Truth* Random House, Inc., New York, 1987.

A companion to a PBS series, with many illustrations. Teaches observation, asking questions, answering questions.

John Pottage *Geometrical Investigations* Addison-Wesley Publishing Company, Inc., Reading, Mass., 1983.

Elizabeth A. Wood *Crystals and Light* D. Van Nostrand Company, Inc., Princeton, New Jersey, 1964.

Elizabeth A. Wood *Crystals—A Handbook for School Teachers* Elizabeth A. Wood, 1972.

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Figure 1

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Figure 2

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Figure 3

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Figure 4

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Figure 5

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Figure 6

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Figure 7

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Figure 8